

ON THE EXISTENCE OF EULER-LAGRANGE ORBITS SATISFYING THE CONORMAL BOUNDARY CONDITIONS.

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ABSTRACT. Let (M, g) be a closed connected Riemannian manifold, $L : TM \rightarrow \mathbb{R}$ be a Tonelli Lagrangian. Given two closed submanifolds $Q_0, Q_1 \subseteq M$ and a real number k , we study the existence of Euler-Lagrange orbits with energy k connecting Q_0 to Q_1 and satisfying suitable boundary conditions, known as *conormal boundary conditions*. We introduce the Mañé critical value which is relevant for this problem and discuss existence results for supercritical and subcritical energies. We also provide counterexamples showing that all the results are sharp.

1. INTRODUCTION

Let (M, g) be a closed connected Riemannian manifold and let $L : TM \rightarrow \mathbb{R}$ be a Tonelli Lagrangian (that is a smooth fiberwise C^2 -strictly convex and superlinear function). The Euler-Lagrange equation, which in local coordinates is given by

$$\frac{d}{dt} \frac{\partial L}{\partial v}(\gamma, \dot{\gamma}) - \frac{\partial L}{\partial q}(\gamma, \dot{\gamma}) = 0,$$

gives rise to a flow on TM , known as the *Euler-Lagrange flow*. The energy function

$$E(q, v) = d_v L(q, v) \cdot v - L(q, v)$$

associated with L is a prime integral of the motion, meaning that it is constant along solutions of the Euler-Lagrange equation. Moreover, E is Tonelli and attains its minimum at $v = 0$; in particular, the energy level sets $E^{-1}(k)$ are compact and invariant under the Euler-Lagrange flow, which therefore turns out to be complete on TM . Here we are interested in the following

Question. *Given two non-empty closed submanifolds $Q_0, Q_1 \subseteq M$, for which $k \in \mathbb{R}$ does there exist an Euler-Lagrange orbit γ with energy k and satisfying the conormal boundary conditions?*

Without loss of generality we may suppose Q_0, Q_1 connected. Recall that an Euler-Lagrange orbit $\gamma : [0, R] \rightarrow M$ is said to satisfy the *conormal boundary conditions* if

$$\begin{cases} \gamma(0) \in Q_0, \gamma(R) \in Q_1, \\ d_v L(\gamma(0), \dot{\gamma}(0)) \Big|_{T_{\gamma(0)} Q_0} = 0, \\ d_v L(\gamma(R), \dot{\gamma}(R)) \Big|_{T_{\gamma(R)} Q_1} = 0. \end{cases} \quad (1.1)$$

In the case of geodesic flows (i.e. when L is just the kinetic energy defined by g), one is simply requiring that γ is a geodesic hitting Q_0 and Q_1 orthogonally. For sake of conciseness, throughout the paper we will call solutions of (1.1) simply *connecting orbits*.

The question above can also be formulated in the Hamiltonian setting. Let $H : T^*M \rightarrow \mathbb{R}$ be the Tonelli Hamiltonian given by the Fenchel dual of L

$$H(q, p) = \max_{v \in T_q M} [\langle p, v \rangle_q - L(q, v)], \quad (1.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between tangent and cotangent bundle. For which $k \in \mathbb{R}$ does $H^{-1}(k)$ carry a Hamiltonian orbit $u : [0, R] \rightarrow T^*M$ with

$$u(0) \in N^*Q_0, \quad u(R) \in N^*Q_1?$$

Date: September 7, 2016.

2000 Mathematics Subject Classification. 37J45, 58E05.

Key words and phrases. Tonelli Lagrangians, conormal bundles, Mañé critical values.

Here, for $i = 0, 1$, N^*Q_i is the *conormal bundle* of Q_i

$$N^*Q_i := \left\{ (q, p) \in T^*M \mid q \in Q_i, T_qQ_i \subseteq \ker p \right\}.$$

We refer to [3], [15] or [21, Section 6.4] for general facts and properties of conormal bundles.

Remark 1.1. *It follows from the Hamiltonian formulation that a necessary condition for the existence of connecting orbits is that the energy level set $H^{-1}(k)$ intersects both the conormal bundles of Q_0 and Q_1 , namely*

$$H^{-1}(k) \cap N^*Q_i \neq \emptyset, \quad \text{for } i = 0, 1. \quad (1.3)$$

We set

$$k(L; Q_0, Q_1) := \inf \left\{ k \in \mathbb{R} \mid (1.3) \text{ holds} \right\}.$$

The above observation can be phrased by saying that $k \geq k(L; Q_0, Q_1)$ is a necessary condition for the existence of connecting orbits. This condition alone is however not sufficient, as we will show in Section 6.

A particular class of Tonelli Lagrangians is given by the so-called *magnetic Lagrangians*, i.e. smooth functions on TM of the form

$$L(q, v) = \frac{1}{2}|v|^2 + \vartheta_q(v), \quad (1.4)$$

where $|\cdot|$ is the norm induced by the Riemannian metric g and ϑ is a smooth one-form on M . The reason for this terminology is that they can be thought of as modelling the motion of a unitary mass and charge particle under the effect of the magnetic field $\sigma = d\vartheta$.

In the Lagrangian setting, condition (1.3) for a magnetic Lagrangian is expressed by

$$k \geq \max \left\{ \min_{q \in Q_0} \frac{1}{2} |\mathcal{P}_0 w_q|^2, \min_{q \in Q_1} \frac{1}{2} |\mathcal{P}_1 w_q|^2 \right\}, \quad (1.5)$$

where $\mathcal{P}_i : TM|_{Q_i} \rightarrow TQ_i$ denotes the orthogonal projection and $w_q \in T_qM$ is the unique tangent vector representing $\vartheta_q \in T_q^*M$. The right-hand side of (1.5) is precisely $k(L; Q_0, Q_1)$; if it is non-zero, then we cannot expect existence of connecting orbits for every positive energy, even if the submanifolds intersect or if $Q_0 = Q_1$.

Remark 1.2. *When $Q_0 = \{q_0\}$ and $Q_1 = \{q_1\}$ are points in M , the question above reduces to the problem of finding those energy levels which contain Euler-Lagrange orbits connecting q_0 and q_1 . This problem has an easy answer when L is a mechanical Lagrangian, i.e. of the form*

$$L(q, v) = \frac{1}{2}|v|^2 - V(q), \quad (1.6)$$

with V smooth function on M (potential energy), but is made extremely hard by the presence of a magnetic potential ϑ (see e.g. [18, Chapter I.3 and Appendix F]). We will get back on this later on in this introduction and in the last section.

Remark 1.3. *The conormal boundary conditions (1.1) make also sense for submanifolds of $M \times M$ which are not necessarily of the form $Q_0 \times Q_1$. In this sense, the problem of finding periodic orbits of the Euler-Lagrange flow can be viewed as the problem of finding Euler-Lagrange orbits satisfying the conormal boundary conditions for $\Delta \subseteq M \times M$ diagonal.*

The key fact that will be exploited throughout this paper is that connecting orbits with energy k correspond to the critical points of the *free-time Lagrangian action functional*

$$\mathbb{A}_k : \mathcal{M}_Q \longrightarrow \mathbb{R}, \quad \mathbb{A}_k(x, T) = T \int_0^1 \left[L\left(x(s), \frac{x'(s)}{T}\right) + k \right] ds,$$

where $Q = Q_0 \times Q_1$ and $\mathcal{M}_Q = H_Q^1([0, 1], M) \times (0, +\infty)$ is the Hilbert manifold of H^1 -paths connecting Q_0 with Q_1 with arbitrary interval of definition. When $Q_0 \cap Q_1 \neq \emptyset$, we identify $Q_0 \cap Q_1$ with the subset of \mathcal{M}_Q made of constant loops with values in $Q_0 \cap Q_1$.

Notice that \mathbb{A}_k is well-defined only under the additional assumption that L is quadratic at infinity; this is however not a problem for our purpose. Indeed, since the energy level $E^{-1}(k)$ is compact,

we can always modify L outside it to achieve the quadratic growth condition. Hereafter all the Lagrangians will be thus supposed without loss of generality to be quadratic at infinity.

The goal of the present work will be to see under which assumptions the existence of critical points for \mathbb{A}_k is guaranteed. It is clear that in this study a crucial role will be played by the analytical (e.g. “compactness” and the Palais-Smale condition) and geometric properties (e.g. boundedness or the presence of a mountain-pass geometry) of \mathbb{A}_k , as well as by the topological properties of the space \mathcal{M}_Q . However, if on the one hand the topology of \mathcal{M}_Q clearly do not depend on k , on the other hand the properties of \mathbb{A}_k change drastically when crossing a suitable energy value. This is actually no surprise, since also the dynamical and geometric properties of the Euler-Lagrange flow change when crossing suitable *Mañé critical values* (cf. [1, 11]).

In general, the critical points for \mathbb{A}_k one might expect to find are either

- global (or local) minimizers,

or

- mountain passes (or more generally minimax critical points).

The questions one has to address are therefore the following

- (1) For which k is \mathbb{A}_k bounded from below on the connected components of \mathcal{M}_Q ? And for those values of k , on which connected components of \mathcal{M}_Q does then \mathbb{A}_k admit minimizers?
- (2) Assume that the topology of \mathcal{M}_Q (or the geometry of \mathbb{A}_k) allows to define a suitable minimax class. For which k does this then yield existence of critical points for \mathbb{A}_k ?

We shall however observe already at this point that a general existence result of critical points for \mathbb{A}_k cannot be obtained since there are examples of Euler-Lagrange flows and of submanifolds Q_0, Q_1 for which there are no connecting orbits. Consider for instance the geodesic flow on the flat Torus (\mathbb{T}^2, g_{flat}) and pick Q_0, Q_1 as in Figure 1 below.

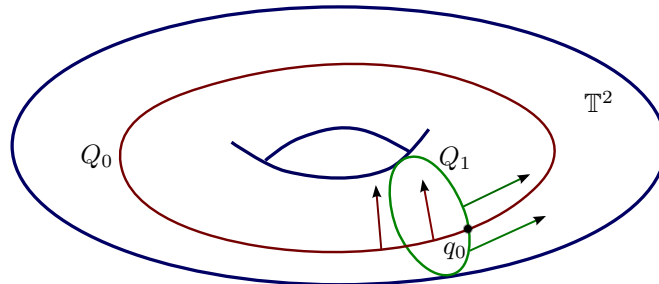


Figure 1. Existence of connecting orbits might fail in full generality.

The conormal boundary conditions (1.1) imply that geodesics connecting Q_0 with Q_1 have to hit both Q_0 and Q_1 orthogonally, which is not possible (except for the constant solution in the intersection point q_0). It follows that for every $k > 0$ there are no geodesics connecting Q_0 with Q_1 and satisfying the conormal boundary conditions. From a variational viewpoint, what goes wrong in this example is that \mathcal{M}_Q is connected, contains constant paths and $\pi_l(\mathcal{M}_Q, \{q\})$ is trivial for every $l \in \mathbb{N}$, for \mathcal{M}_Q is contractible. This implies that \mathbb{A}_k has infimum zero on \mathcal{M}_Q , for every $k > 0$, and this is not attained. Also, one cannot expect to find minimax critical points, since there are no non-trivial minimax classes to play with.

This is however (under some mild assumption on the intersection if $Q_0 \cap Q_1 \neq \emptyset$) the only possible counterexample, at least if k is “sufficiently large” as we now briefly explain. Let $G < \pi_1(M)$ be the smallest normal subgroup containing both $\iota_*(\pi_1(Q_0))$ and $\iota_*(\pi_1(Q_1))$, where $\iota : Q_i \hookrightarrow M$ is the canonical inclusion; consider the cover $M' = \widetilde{M}/G$, with \widetilde{M} universal cover of M , and define

$$c(L; Q_0, Q_1) := \inf_{u \in C^\infty(M')} \sup_{q \in M'} H'(q, d_q u), \quad (1.7)$$

where H' is the lift to M' of the Hamiltonian $H : T^*M \rightarrow \mathbb{R}$ associated with L .

Remark 1.4. We have $c(L; Q_0, Q_1) \geq k(L; Q_0, Q_1)$ (this will be proved in Proposition 3.5). Moreover, if $Q_0 = \{q_0\}$, $Q_1 = \{q_1\}$, then

$$k(L; \{q_0\}, \{q_1\}) = \max\{E(q_1, 0), E(q_2, 0)\}, \quad c(L; \{q_0\}, \{q_1\}) = c_u(L).$$

Here $c_u(L)$ is the Mañé critical value of the universal cover and is defined as in (1.7) replacing M' with the universal cover of M .

In Section 3 we will show that, for $k \geq c(L; Q_0, Q_1)$, \mathbb{A}_k is bounded from below on each connected component of \mathcal{M}_Q and it is unbounded from below on each connected component otherwise. Furthermore, for $k > c(L; Q_0, Q_1)$ every Palais-Smale sequence for \mathbb{A}_k with times bounded away from zero admits converging subsequences. These facts will allow us in Section 4 to prove the following:

Theorem 1. Let \mathcal{N} be a connected component of \mathcal{M}_Q ; then we have:

- (1) If \mathcal{N} does not contain constant paths, then for all $k > c(L; Q_0, Q_1)$ there exists a global minimizer of $\mathbb{A}_k|_{\mathcal{N}}$.
- (2) Suppose now that \mathcal{N} contains constant paths and define

$$k_{\mathcal{N}}(L) := \sup \left\{ k \in \mathbb{R} \mid \inf_{\mathcal{N}} \mathbb{A}_k < 0 \right\} \in [c(L; Q_0, Q_1), +\infty).$$

Then the following hold:

- (a) For all $k \in (c(L; Q_0, Q_1), k_{\mathcal{N}}(L))$ there is a global minimizer of $\mathbb{A}_k|_{\mathcal{N}}$.
- (b) If $Q_0 \cap Q_1$ is connected, \mathcal{N} has the retraction property (see the beginning of Section 4 for the definition), and $\pi_l(\mathcal{N}, Q_0 \cap Q_1) \neq 0$ for some $l \geq 1$, then for all $k > k_{\mathcal{N}}(L)$ there is a minimax critical point for $\mathbb{A}_k|_{\mathcal{N}}$.
- (c) If $Q_0 \cap Q_1$ is not connected and at least one of its connected component is isolated, then for all $k > k_{\mathcal{N}}(L)$ there exists a minimax critical point for $\mathbb{A}_k|_{\mathcal{N}}$.

In the following we refer to *supercritical energies* whenever $k > c(L; Q_0, Q_1)$ and to *subcritical energies* whenever $k \in (k(L; Q_0, Q_1), c(L; Q_0, Q_1))$.

A particular case of intersecting submanifolds is given by the choice $Q_0 = Q_1$, which corresponds to (a particular case of) the *Arnold chord conjecture* about the existence of a Reeb orbit starting and ending at a given Legendrian submanifold of a contact manifold, see [4, 25], but in a possibly *virtually* contact situation (c.f. introduction of [13] for the definition), since in general $c(L; Q_0, Q_0)$ might be strictly lower than $c_0(L)$. Recall indeed that energy levels above $c_u(L)$ are virtually contact (cf. [13, Lemma 5.1]), however they are known to be not of contact type if $c_u(L) < k < c_0(L)$ (cf. [11, Proposition B.1]). Here $c_0(L)$ is defined as in (1.7) replacing M' with the abelian cover of M and is called the *Mañé critical value of the abelian cover*.

In our setting an *Arnold chord* is simply an Euler-Lagrange orbits starting and ending at Q_0 and satisfying the conormal boundary conditions.

Corollary 1. Let $Q_0 \subseteq M$ be a non-empty closed connected submanifold and define $c(L; Q_0)$ as in (1.7) just by setting $Q_0 = Q_1$. Then the following hold:

- (1) For every $k > c(L; Q_0)$ and for every connected component of \mathcal{M}_Q that does not contain constant paths there exists an Arnold chord with energy k which is a global minimizer of \mathbb{A}_k among its connected component.
- (2) Let \mathcal{N} be the connected component of \mathcal{M}_Q containing the constant paths. For every $k \in (c(L; Q_0), k_{\mathcal{N}}(L))$ there is an Arnold chord with energy k which is a global minimizer of \mathbb{A}_k on \mathcal{N} . Moreover, if $\pi_l(M, Q_0) \neq \{0\}$ for some $l \geq 2$, then for every $k > k_{\mathcal{N}}(L)$ there exists an Arnold chord in \mathcal{N} with energy k .

In particular, if $Q_0 \neq M$, then for all $k > c(L; Q_0)$, $k \neq k_{\mathcal{N}}(L)$, there is an Arnold chord with energy k .

Existence results for subcritical energies are harder to achieve than the corresponding ones for supercritical energies and the reason for that are of various nature.

First, when $k < c(L; Q_0, Q_1)$, \mathbb{A}_k could have Palais-Smale sequences with times going to infinity. In fact, the lack of the Palais-Smale condition for subcritical energies is ultimately responsible for the fact that one gets existence results which hold only for *almost every energy* in a suitable range of subcritical energies.

Second, in case the intersection $Q_0 \cap Q_1$ is empty the problem might have no solutions for every $k \in (k(L; Q_0, Q_1), c(L; Q_0, Q_1))$ as it contains, as a very special case, the problem of finding the energy levels for which any two points in M can be joined by an Euler-Lagrange orbit. In this direction it has been proven by Mañé in [23, Page 151] that, for every $k > c_0(L)$, every pair of points in M can be joined by an Euler-Lagrange orbit. This result has been strenghtened by Contreras in [11] to every $k > c_u(L)$. In Section 6 we provide an example showing that Contreras' result, and Theorem 1 as well, are actually sharp.

Theorem 2. *There exist a Tonelli Lagrangian $L : T\mathbb{T}^2 \rightarrow \mathbb{R}$ and two disjoint submanifolds $Q_0, Q_1 \subseteq \mathbb{T}^2$ such that $c_u(L) < c(L; Q_0, Q_1)$ and with no connecting orbits having energy $k \leq c(L; Q_0, Q_1)$. Moreover, there are points $q_0 \in Q_0$ and $q_1 \in Q_1$ such that there are no Euler-Lagrange orbits connecting them with energy $k \leq c_u(L)$.*

In order to get existence results for subcritical energies one has therefore to assume that $Q_0 \cap Q_1 \neq \emptyset$. Denote with \mathcal{N} the connected component of \mathcal{M}_Q containing the constant paths and with Ω an isolated connected component of $Q_0 \cap Q_1$, meaning that there exists $\epsilon > 0$ such that $B_\epsilon(\Omega)$ is disjoint from any other connected components of $Q_0 \cap Q_1$. Now set

$$k_\Omega := \min \left\{ c(L; Q_0, Q_1), \max_{q \in \Omega} E(q, 0) + \lambda \cdot \max_{q \in \Omega} |d_v L(q, 0)|^2 \right\},$$

where $\lambda > 0$ is a constant depending only on L which equals $\frac{1}{2}$ in case L is a magnetic Lagrangian (see Section 5 for the precise definition). The definition of k_Ω in case Ω is not isolated is more delicate and will be postponed to Section 5.

Remark 1.5. *The energy values $k(L; Q_0, Q_1)$ and k_Ω strongly depend on how the submanifolds Q_0 and Q_1 sit inside M . This is in sharp contrast with what happens for the critical value $c(L; Q_0, Q_1)$ which only depends on the homotopy classes of (M, Q_0) and (M, Q_1) , meaning that if we have a continuous map $\{F_t : (M, Q_0^0) \rightarrow (M, Q_1^t)\}_{t \in I}$, then*

$$c(L; Q_0^0, Q_1) = c(L; Q_0^t, Q_1), \quad \forall t \in I.$$

Moreover it follows directly from the definition that $c(L; Q_0, Q_1) = c(L; Q_1, Q_0)$.

In general the energy value k_Ω need not coincide with $c(L; Q_0, Q_1)$; examples will be provided in Section 6. The relevance of k_Ω relies on the fact that, for $k \in (k_\Omega, c(L; Q_0, Q_1))$ the free-time action functional has a mountain-pass geometry on \mathcal{N} . The two valleys are represented by the set of constant paths and by the set of paths with negative action (which is non-empty as $k < c(L; Q_0, Q_1)$). Exploiting this mountain-pass geometry we get the following

Theorem 3. *For almost every $k \in (\inf_\Omega k_\Omega, c(L; Q_0, Q_1))$ there is a connecting orbit with energy k .*

In the theorem above, by taking the infimum of k_Ω over all connected components Ω of $Q_0 \cap Q_1$ we get a critical value which is a priori smaller than $k_{Q_0 \cap Q_1}$ and, hence, a sharper result.

The “almost every” relies exactly on the lack of the Palais-Smale condition for \mathbb{A}_k for subcritical energies. To overcome this difficulty one has to use an argument originally due to Struwe [27], which has already been intensively applied to the existence of periodic Euler-Lagrange orbits for subcritical energies [1, 2, 6, 7, 11], called the *Struwe monotonicity argument*.

As a trivial corollary we get the following existence result of Arnold chords for subcritical energies.

Corollary 2. *Let $Q_0 \subseteq M$ be a non-empty closed connected submanifold. Then for almost every $k \in (k_{Q_0}, c(L; Q_0))$ there is an Arnold chord with energy k .*

As we will see in Section 6, also Theorem 3 above is sharp.

Theorem 4. *For every closed surface Σ , there exist a Tonelli Lagrangian $L : T\Sigma \rightarrow \mathbb{R}$ and intersecting submanifolds $Q_0, Q_1 \subseteq \Sigma$ such that*

$$k(L; Q_0, Q_1) < k_\Omega < c(L; Q_0, Q_1)$$

and with no connecting orbits having energy $k < k_\Omega$.

We end this introduction giving a brief summary of the contents of the paper:

- In Section 2 we introduce the free-time Lagrangian action functional \mathbb{A}_k rigorously and discuss its properties (with particular attention to the Palais-Smale condition and to the completeness of the negative gradient flow).
- In Section 3 we define the Mañé critical value $c(L; Q_0, Q_1)$ which is relevant for the problem and show how the properties of \mathbb{A}_k change when considering “subcritical” rather than “supercritical” values of k .
- In Section 4 we deal with the case of supercritical energies and prove Theorem 1.
- In Section 5 we consider the case of subcritical energies and prove Theorem 3.
- Finally, in Section 6, we prove Theorems 2 and 4.

2. THE FREE-TIME LAGRANGIAN ACTION FUNCTIONAL

For any given absolutely continuous curve $\gamma : [0, T] \rightarrow M$ we define $x : [0, 1] \rightarrow M$ as $x(s) := \gamma(sT)$. Throughout the whole work we will identify γ with the pair (x, T) .

To avoid confusion we will always denote with a *dot* the derivative with respect to t and with a *prime* the derivative with respect to s .

Fix a real number k , the value of the energy for which we would like to find Euler-Lagrange orbits satisfying the conormal boundary conditions (1.1). Recall that, since the energy level $E^{-1}(k)$ is compact, up to the modification of L outside it, we may assume the Tonelli Lagrangian L to be quadratic at infinity. In particular

$$L(q, v) \geq a|v|^2 - b, \quad \forall (q, v) \in TM, \quad (2.1)$$

$$d_{vv}L(q, v)[u, u] \geq 2a|u|^2, \quad \forall (q, v) \in TM, \quad \forall u \in T_qM, \quad (2.2)$$

for suitable numbers $a > 0$, $b \in \mathbb{R}$ and

$$S_k(x, T) := \int_0^T \left[L(\gamma(t), \dot{\gamma}(t)) + k \right] dt = T \int_0^1 \left[L\left(x(s), \frac{x'(s)}{T}\right) + k \right] ds \quad (2.3)$$

is well-defined for every $x \in H^1([0, 1], M)$. Hence, we get a well-defined functional

$$S_k : H^1([0, 1], M) \times (0, +\infty) \longrightarrow \mathbb{R},$$

called the *free-time action functional*. The domain of definition $\mathcal{M} := H^1([0, 1], M) \times (0, +\infty)$ of S_k can be interpreted as the space of H^1 -paths in M with arbitrary interval of definition through the identification $\gamma = (x, T)$ above and it has a natural structure of product Hilbert manifold given by the product metric

$$g_{\mathcal{M}} := g_{H^1} + dT^2, \quad (2.4)$$

where g_{H^1} is the standard metric on $H^1([0, 1], M)$ induced by the given Riemannian metric g on M (see [3] for further details). Obviously, $(\mathcal{M}, g_{\mathcal{M}})$ is not complete as the factor $(0, +\infty)$ is not complete with respect to the Euclidean metric. The following proposition is about the regularity of the free-time action functional S_k ; for the proof we refer again to [3] (see also [5, Proposition 3.1.1]).

Proposition 2.1. *The following hold:*

- (1) $S_k \in C^{1,1}(\mathcal{M})$ and it has second Gateaux differential at every point.
- (2) S_k is twice Fréchet differentiable at every point if and only if L is electromagnetic on the whole TM ; in this case, S_k is actually smooth.

Let now $Q_0, Q_1 \subseteq M$ be non-empty closed connected submanifolds. Since we want to prove the existence of connecting orbits, we shall consider the restriction of S_k to the smooth submanifold

$$\mathcal{M}_Q := H_Q^1([0, 1], M) \times (0, +\infty),$$

where $Q = Q_0 \times Q_1$ and $H_Q^1([0, 1], M)$ is the space of H^1 -paths $x : [0, 1] \rightarrow M$ connecting Q_0 with Q_1 . We denote with \mathbb{A}_k the restriction $S_k|_{\mathcal{M}_Q}$. The importance of \mathbb{A}_k relies on the following:

Proposition 2.2. *A curve $\gamma = (x, T)$ is a connecting orbit with energy $E(\gamma, \dot{\gamma}) = k$ if and only if (x, T) is a critical point of the free-time action functional \mathbb{A}_k .*

Proof. The pair (x, T) is a critical point for \mathbb{A}_k if and only if

$$d\mathbb{A}_k(x, T)[(\zeta, H)] = 0$$

for any choice of (ζ, H) . It is well-known (see e.g. [3]) that the condition

$$d_x \mathbb{A}_k(x, T)[(\zeta, 0)] = 0$$

is equivalent to $\gamma(t) := x(t/T) \in H_Q^1([0, T], M)$ being an Euler-Lagrange orbit satisfying the conormal boundary conditions (1.1). Furthermore, a simple computation shows that

$$\frac{\partial \mathbb{A}_k}{\partial T}(x, T) = \frac{1}{T} \int_0^T [k - E(\gamma(t), \dot{\gamma}(t))] dt \quad (2.5)$$

which implies that $E(\gamma, \dot{\gamma}) = k$, since the energy is constant along γ . \square

Completeness properties for \mathbb{A}_k . Since the Hilbert manifold \mathcal{M}_Q is not complete, it is useful to know whether sublevel sets of the free-time action functional \mathbb{A}_k are complete or not. With the next lemma we see that completeness on a given connected component \mathcal{N} of \mathcal{M}_Q only depends on the fact that \mathcal{N} contains constant paths or not.

Lemma 2.3. *The following statements hold:*

- (1) *The sublevel sets of \mathbb{A}_k in each connected component \mathcal{N} of \mathcal{M}_Q not containing constant paths are complete.*
- (2) *If (x_h, T_h) is such that $T_h \rightarrow 0$, then*

$$\liminf_{h \rightarrow +\infty} \mathbb{A}_k(x_h, T_h) \geq 0. \quad (2.6)$$

Proof. By (2.1) we have the chain of inequalities

$$\begin{aligned} \mathbb{A}_k(x, T) &= T \int_0^1 \left[L\left(x(s), \frac{x'(s)}{T}\right) + k \right] ds \\ &\geq T \int_0^1 \left[a \frac{|x'(s)|^2}{T^2} - b + k \right] ds \\ &= \frac{a}{T} \int_0^1 |x'(s)|^2 ds + T(k - b) \\ &\geq \frac{a}{T} l(x)^2 + T(k - b) \end{aligned} \quad (2.7)$$

where $l(x)$ denotes the length of the path x . Since \mathcal{N} does not contain constant paths, the length of any path in \mathcal{N} is bounded away from zero by a suitable positive constant. Therefore, T is bounded away from zero on

$$\left\{ (x, T) \in \mathcal{N} \mid \mathbb{A}_k(x, T) \leq c \right\}$$

for any $c \in \mathbb{R}$, proving the statement.

Inequality (2.7) actually also proves the second statement. In fact, if $T_h \rightarrow 0$ then

$$T_h(k - b) \rightarrow 0, \quad \text{for } h \rightarrow +\infty$$

and hence the action $\mathbb{A}_k(x_h, T_h)$ is eventually bigger than $-\epsilon$, for arbitrary $\epsilon > 0$. \square

Corollary 2.4. *If $c < 0$, then the sublevel set $\{\mathbb{A}_k \leq c\}$ is complete.*

Proof. Follows directly from Statement 2 in Lemma 2.3. \square

We end this section studying the possible sources of non-completeness of the negative gradient flow of \mathbb{A}_k on \mathcal{M}_Q . Up to changing $-\nabla \mathbb{A}_k$ with the conformally equivalent bounded vector field

$$-\frac{\nabla \mathbb{A}_k}{\sqrt{1 + |\nabla \mathbb{A}_k|^2}}$$

we may assume the negative gradient flow to be complete on every connected component of \mathcal{M}_Q not containing constant paths. Also, on the connected components containing constant paths, incompleteness occurs only if there are flow-lines for which $T(\cdot) \rightarrow 0$ in finite time. The next lemma ensures that, for such flow lines, \mathbb{A}_k necessarily goes to zero.

Lemma 2.5. *Let $(x(\cdot), T(\cdot)) : [0, \sigma^*) \rightarrow \mathcal{M}_Q$ be a negative gradient flow-line with*

$$\liminf_{\sigma \rightarrow \sigma^*} T(\sigma) = 0.$$

Then

$$\lim_{\sigma \rightarrow \sigma^*} \mathbb{A}_k(x(\sigma), T(\sigma)) = 0.$$

Proof. The proof is analogous to the one of Lemma 3.3 in [1], where the case of periodic orbits is considered. Since both E and L are quadratic at infinity, we have

$$E(q, v) \geq c_0 L(q, v) - c_1$$

for some $c_0 > 0$ and $c_1 \in \mathbb{R}$. Therefore from (2.5) it follows that

$$\begin{aligned} \frac{\partial \mathbb{A}_k}{\partial T}(x, T) &= \frac{1}{T} \int_0^T \left[k - E(\gamma(t), \dot{\gamma}(t)) \right] dt \\ &\leq \frac{1}{T} \int_0^T \left[k - c_0 L(\gamma(t), \dot{\gamma}(t)) + c_1 \right] dt \\ &= (c_0 + 1)k + c_1 - \frac{c_0}{T} \mathbb{A}_k(x, T) \end{aligned}$$

and hence

$$\mathbb{A}_k(x, T) \leq \frac{T}{c_0} \left[(c_0 + 1)k + c_1 - \frac{\partial \mathbb{A}_k}{\partial T}(x, T) \right] = \frac{T}{c_0} \left[C - \frac{\partial \mathbb{A}_k}{\partial T}(x, T) \right], \quad (2.8)$$

where C is a suitable constant. By assumption, there is a sequence $\sigma_h \uparrow \sigma^*$ with

$$T'(\sigma_h) \leq 0, \quad T(\sigma_h) \rightarrow 0.$$

Since $\sigma \mapsto (x(\sigma), T(\sigma))$ is a negative gradient flow-line, we have

$$0 \geq T'(\sigma_h) = -\frac{\partial \mathbb{A}_k}{\partial T}(x(\sigma_h), T(\sigma_h))$$

and hence

$$\mathbb{A}_k(x(\sigma_h), T(\sigma_h)) \leq \frac{T(\sigma_h)}{c_0} \left[C - \frac{\partial \mathbb{A}_k}{\partial T}(x(\sigma_h), T(\sigma_h)) \right] \leq \frac{C}{c_0} T(\sigma_h).$$

Since $T(\sigma_h) \rightarrow 0$, from the inequality above we deduce that

$$\limsup_{h \rightarrow +\infty} \mathbb{A}_k(x(\sigma_h), T(\sigma_h)) \leq 0.$$

The assertion follows now from Statement 2 in Lemma 2.3 and from the monotonicity of the function $\sigma \mapsto \mathbb{A}_k(x(\sigma), T(\sigma))$. \square

The Palais-Smale condition for \mathbb{A}_k . Recall that a Palais-Smale sequence at level c for \mathbb{A}_k is a sequence $(x_h, T_h) \subseteq \mathcal{M}_Q$ such that

$$\mathbb{A}_k(x_h, T_h) \rightarrow c, \quad |d\mathbb{A}_k(x_h, T_h)| \rightarrow 0,$$

where $|\cdot|$ denotes the norm on $T^*\mathcal{M}_Q$ induced by the Riemannian metric $g_{\mathcal{M}}$ in (2.4).

When looking for critical points of \mathbb{A}_k (and more generally of a given functional defined on a Hilbert manifold) it is natural to consider Palais-Smale sequences as a “source of critical points”, since their limit points are by definition critical points. However, it is in general not true that Palais-Smale sequences have limit points. Therefore, it is worth looking for necessary and sufficient conditions for a Palais-Smale sequence to admit converging subsequences. Palais-Smale sequences with times going to zero surely do not possess limit points. However, they might occur only in connected components of \mathcal{M}_Q that contain constant paths, as Lemma 2.3 shows. The next lemma ensures also that such Palais-Smale sequences may appear only at level zero.

Lemma 2.6. *Let $\gamma_h = (x_h, T_h)$ be a Palais-Smale sequence at level $c \in \mathbb{R}$ for \mathbb{A}_k such that $T_h \rightarrow 0$. Then necessarily $c = 0$.*

Proof. First we prove that

$$\int_0^{T_h} |\dot{\gamma}_h(t)|^2 dt = O(T_h), \quad \text{for } h \rightarrow +\infty. \quad (2.9)$$

Being (x_h, T_h) a Palais-Smale sequence for \mathbb{A}_k , we have

$$|d\mathbb{A}_k(x_h, T_h)| = o(1), \quad \text{for } h \rightarrow +\infty.$$

In particular, using (2.5) we get that

$$\left| d\mathbb{A}_k(x_h, T_h) \left[\frac{\partial}{\partial T} \right] \right| = \left| \frac{\partial \mathbb{A}_k}{\partial T}(x_h, T_h) \right| = \left| \frac{1}{T_h} \int_0^{T_h} [E(\gamma_h(t), \dot{\gamma}_h(t)) - k] dt \right| = o(1)$$

and hence

$$\alpha_h := \frac{1}{T_h} \int_0^{T_h} [E(\gamma_h(t), \dot{\gamma}_h(t)) - k] dt \longrightarrow 0. \quad (2.10)$$

Now by assumption E is quadratic at infinity and hence $E(q, v) \geq a'|v|^2 - b'$, for some $a' > 0$ and $b' \in \mathbb{R}$. Using this in (2.10) we get that

$$\alpha_h = \frac{1}{T_h} \int_0^{T_h} [E(\gamma_h(t), \dot{\gamma}_h(t)) - k] dt \geq \frac{1}{T_h} \int_0^{T_h} [a' |\dot{\gamma}_h(t)|^2 - b' - k] dt$$

and hence

$$\int_0^{T_h} |\dot{\gamma}_h(t)|^2 dt \leq \frac{T_h}{a'} [\alpha_h + b' + k],$$

which implies (2.9). Since also L is quadratic at infinity we have

$$a|v|^2 - b \leq L(q, v) \leq \tilde{a}|v|^2 + \tilde{b}$$

for some constants $a, \tilde{a} > 0$ and $b, \tilde{b} \in \mathbb{R}$. The first inequality implies

$$\begin{aligned} \mathbb{A}_k(x_h, T_h) &= \int_0^{T_h} [L(\gamma_h(t), \dot{\gamma}_h(t)) + k] dt \\ &\geq a \int_0^{T_h} |\dot{\gamma}_h(t)|^2 dt + T_h(k - b) = O(T_h) \end{aligned}$$

while the second yields

$$\begin{aligned} \mathbb{A}_k(x_h, T_h) &= \int_0^{T_h} [L(\gamma_h(t), \dot{\gamma}_h(t)) + k] dt \\ &\leq \tilde{a} \int_0^{T_h} |\dot{\gamma}_h(t)|^2 dt + T_h(k + \tilde{b}) = O(T_h) \end{aligned}$$

and hence obviously $\mathbb{A}_k(x_h, T_h) \rightarrow 0$. □

The following lemma ensures the existence of converging subsequences for any Palais-Smale sequence with times bounded and bounded away from zero. The proof is analogous (with some minor adjustments) to the one of [11, Proposition 3.12] (or [1, Lemma 5.3]), where the case of periodic orbits is considered; see [5, Lemma 3.2.2] for the details. This lemma combined with Lemma 2.6 above shows that the only Palais-Smale sequences at level $c \neq 0$ which may cause troubles are those for which the times diverge.

Lemma 2.7. *Let (x_h, T_h) be a Palais-Smale sequence at level $c \in \mathbb{R}$ for \mathbb{A}_k in some connected component of \mathcal{M}_Q with $0 < T_* \leq T_h \leq T^* < +\infty$. Then, (x_h, T_h) is compact in \mathcal{M}_Q , meaning that it admits a converging subsequence.*

3. THE MAÑÉ CRITICAL VALUE $c(L; Q_0, Q_1)$

The following numbers should be interpreted as energy levels and mark important dynamical and geometric changes for the Euler-Lagrange flow induced by the Tonelli Lagrangian L . The reader may take a look at [1] or [12] for a survey on the relevance of these energy values and on their relation with the geometric and dynamical properties of the Euler-Lagrange flow. First, let us define the *Mañé critical value* associated to L as

$$c(L) := \inf \left\{ k \in \mathbb{R} \mid S_k(\gamma) \geq 0, \forall \gamma \text{ closed loop} \right\}. \quad (3.1)$$

Second, we recall the definition of the *Mañé critical value of the Abelian cover*

$$c_0(L) := \inf \left\{ k \in \mathbb{R} \mid S_k(\gamma) \geq 0, \forall \gamma \text{ closed loop homologous to zero} \right\}. \quad (3.2)$$

This is the relevant energy value, for instance, when trying to use methods coming from Finsler geometry. Indeed, for every $k > c_0(L)$ the Euler-Lagrange flow restricted to the energy level $E^{-1}(k)$ is conjugated to the geodesic flow defined by a suitable Finsler metric (see [1] for the details). On the other hand, for exact magnetic flows (i.e. Euler-Lagrange flows associated with magnetic Lagrangians) on surfaces, if $k < c_0(L)$ then there exist periodic orbits with energy k which are local minimizers of the free-period Lagrangian action functional, as explained in [2] and in [14]. One would be tempted to say that, at least on surfaces and for a suitable range of energies, a similar existence result of local minimizers for the free-time action functional \mathbb{A}_k should hold also in our setting; this is unfortunately not the case, as we will see in Section 6.

When looking for periodic orbits, the energy value which turns out to be relevant for the properties of the free-period action functional (see again [1] or [11]) is however the so-called *Mañé critical value of the universal cover*

$$c_u(L) := \inf \left\{ k \in \mathbb{R} \mid S_k(\gamma) \geq 0, \forall \gamma \text{ closed contractible loop} \right\}. \quad (3.3)$$

We also define

$$e_0(L) := \max_{q \in M} E(q, 0) \quad (3.4)$$

to be the maximum of the energy on the zero section of TM . The topology of the energy level sets changes when crossing the value $e_0(L)$. In fact, for any $k > e_0(L)$, the energy level sets $E^{-1}(k)$ have all the same topology, namely of a sphere bundle over M . This is instead false for $k < e_0(L)$, being the projection $E^{-1}(k) \rightarrow M$ not surjective any more. Notice that

$$\min E \leq e_0(L) \leq c_u(L) \leq c_0(L) \leq c(L). \quad (3.5)$$

If L is a magnetic Lagrangian as in (1.4), then $\min E = e_0(L) = 0$. When the magnetic potential ϑ vanishes we additionally have

$$0 = \min E = e_0(L) = c_u(L) = c_0(L) = c(L),$$

but in general the inequalities in (3.5) are strict. See for instance [24, Page 151] (or Section 6) for an example where $e_0(L) < c_0(L) < c(L)$ and [26] for an example where $c_u(L) < c_0(L)$. The values $c_u(L)$ and $c_0(L)$ clearly coincide when $\pi_1(M)$ is abelian; more generally, they coincide whenever $\pi_1(M)$ is amenable (cf. [16]).

When the fundamental group of M is rich, there are other Mañé critical values, which are associated to the different covering spaces of M . We now show which one is relevant for our purposes. Given a covering map $p : M_1 \rightarrow M$, consider the lifted Lagrangian

$$L_1 := dp \circ L : TM_1 \rightarrow \mathbb{R}$$

and the associated critical value $c(L_1)$ as in (3.1). The following lemma is straightforward.

Lemma 3.1. *There holds $c(L_1) \leq c(L)$. If p is a finite covering, then $c(L_1) = c(L)$.*

Remark 3.2. *Mañé critical values have also an equivalent Hamiltonian definition (see for instance [10] or [12]). As above let M_1 be a cover of M and denote by H_1 the lift of the Tonelli Hamiltonian H associated with L to the cover M_1 ; then there holds*

$$c(L_1) = \inf_{u \in C^\infty(M_1)} \sup_{q \in M_1} H_1(q, d_q u). \quad (3.6)$$

It is well known that regular covering spaces correspond to normal subgroups of $\pi_1(M)$, i.e. for any regular covering $p : M_1 \rightarrow M$ there is a unique normal subgroup $G < \pi_1(M)$ with

$$M_1 \cong \widetilde{M} / G,$$

where \widetilde{M} denotes the universal cover of M . We denote the Mañé critical value $c(L_1)$ of the lifted Lagrangian by

$$c(L; G) := c(L_1).$$

Lemma 3.3. *Let $G, G' < \pi_1(M)$ be two normal subgroups; then*

$$c(L; \langle G, G' \rangle) = \max \{c(L; G), c(L; G')\},$$

where $\langle G, G' \rangle$ denotes the (normal) subgroup generated by G and G' .

Proof. Since $G < \langle G, G' \rangle$ is a normal subgroup, we have a covering

$$p : \widetilde{M} / G \longrightarrow \widetilde{M} / \langle G, G' \rangle$$

and hence, by Lemma 3.1, $c(L; G) \leq c(L; \langle G, G' \rangle)$. The same holds clearly also when considering G' instead of G and hence we get

$$\max \{c(L; G), c(L; G')\} \leq c(L; \langle G, G' \rangle).$$

Conversely, let $k < c(L; \langle G, G' \rangle)$. By definition there exists

$$\gamma = \alpha_1 \# \beta_1 \# \dots \# \alpha_n \# \beta_n$$

with $\alpha_i \in G$, $\beta_i \in G'$ for all $i = 1, \dots, n$, such that $S_k(\gamma) < 0$. It follows

$$S_k(\gamma) = S_k(\alpha_1) + S_k(\beta_1) + \dots + S_k(\alpha_n) + S_k(\beta_n) < 0.$$

In particular there is one loop, say α_1 , such that $S_k(\alpha_1) < 0$; hence, by definition we have $k < c(L; G)$. This implies the opposite inequality. \square

We want now to understand when the action functional \mathbb{A}_k is bounded from below on each connected component of \mathcal{M}_Q . Thus, let $q_0 \in Q_0$, $q_1 \in Q_1$ and denote by

$$G_0 := \langle \iota_*(\pi_1(Q_0, q_0)) \rangle, \quad G_1 := \langle \iota_*(\pi_1(Q_1, q_1)) \rangle \quad (3.7)$$

the smallest normal subgroups in $\pi_1(M)$ which contain $\iota_*(\pi_1(Q_0))$, $\iota_*(\pi_1(Q_1))$ respectively, where $\iota : Q_0 \hookrightarrow M$, $\iota : Q_1 \rightarrow M$ are the inclusion maps.

Suppose that there exists a loop δ freely-homotopic to an element in $\iota_*(\pi_1(Q_0, q_0))$ and with $S_k(\delta) < 0$. Under this assumption we want to show that \mathbb{A}_k is unbounded from below on every connected component of \mathcal{M}_Q . Without loss of generality we may assume that $\delta \in \iota_*(\pi_1(Q_0, q_0))$, as otherwise we can choose any path ν from q_0 to $\delta(0)$, $\eta \in \pi_1(M, Q_0)$ such that $\eta^{-1} \# \nu^{-1} \# \delta \# \nu \# \eta \in \iota_*(\pi_1(Q_0, q_0))$, and $n \in \mathbb{N}$ large enough such that

$$S_k(\eta^{-1} \# \nu^{-1} \# \delta^n \# \nu \# \eta) < 0.$$

Now fix $\sigma \in \mathcal{M}_Q$. Since Q_0 is connected there exists a path $\mu : [0, 1] \rightarrow Q_0$ such that $(\mu^{-1} \# \sigma \# \mu)(0) = q_0$. Furthermore, we can choose μ in such a way that

$$\mathbb{A}_k(\mu^{-1} \# \sigma \# \mu) = S_k(\mu^{-1}) + \mathbb{A}_k(\sigma) + S_k(\mu) \leq \mathbb{A}_k(\sigma) + c,$$

where c is some constant independent of σ . Therefore, up to adding a uniformly bounded quantity to $\mathbb{A}_k(\sigma)$, we can assume without loss of generality that $\sigma(0) = q_0$. We claim that, for all $n \in \mathbb{N}$, the path $\sigma \# \delta^n$ lies in the same connected component of σ . Indeed, there exists $\alpha \in \pi_1(Q_0, q_0)$ such that $\iota \circ \alpha$ is homotopic to δ with base point q_0 fixed; in particular $\sigma \# \delta \sim \sigma \# (\iota \circ \alpha)$ and now it is easy to see that $\sigma \# (\iota \circ \alpha) \sim \sigma$. A homotopy is for instance given by

$$F : [0, 1] \times [0, 1] \rightarrow M, \quad F(s, \cdot) := \sigma \# (\iota \circ \alpha)|_{[s, 1]}.$$

Since $\mathbb{A}_k(\sigma \# \delta^n) \rightarrow -\infty$ as $n \rightarrow +\infty$, we may conclude that, if such a loop δ exists then the free-time action functional \mathbb{A}_k is unbounded from below on each connected component of \mathcal{M}_Q . In other words, \mathbb{A}_k is unbounded from below on each connected component of \mathcal{M}_Q if $k < c(L; G_0)$ (observe indeed that, by definition, for every $k < c(L; G_0)$ there exists a loop δ satisfying the requirements).

Clearly the same holds when considering Q_1 instead of Q_0 . Therefore we define the *Mañé critical value of the pair Q_0, Q_1* as

$$c(L; Q_0, Q_1) := c(L; \langle G_0, G_1 \rangle) = \max \{c(L; G_0), c(L; G_1)\}. \quad (3.8)$$

We can sum up the discussion above in the following

Lemma 3.4. *For every $k < c(L; Q_0, Q_1)$, the free-time action functional \mathbb{A}_k is unbounded from below on each connected component of \mathcal{M}_Q .*

Proposition 3.5. *We have $k(L; Q_0, Q_1) \leq c(L; Q_0, Q_1)$.*

Proof. We prove this rigorously in case L is a magnetic Lagrangian and then show how to adjust the proof in the general case. Recall that, if $L(q, v) = \frac{1}{2}|v|_q^2 + \vartheta_q(v)$, then

$$k(L; Q_0, Q_1) = \max \left\{ \min_{q \in Q_0} \frac{1}{2} |\mathcal{P}_0 w_q|^2, \min_{q \in Q_1} \frac{1}{2} |\mathcal{P}_1 w_q|^2 \right\},$$

where $\mathcal{P}_i : TM|_{Q_i} \rightarrow TQ_i$ is the orthogonal projection and $w_q \in T_q M$ is the unique vector representing $\vartheta_q \in T_q^* M$. Suppose without loss of generality that

$$k(L; Q_0, Q_1) = \min_{q \in Q_0} \frac{1}{2} |\mathcal{P}_0 w_q|^2 > 0$$

and fix $k < k(L; Q_0, Q_1)$ (if $k(L; Q_0, Q_1) = 0$ then there is nothing to prove, for we trivially have $c(L; Q_0, Q_1) \geq 0$). We now prove that there exists $\delta \in \iota_*(\pi_1(Q_0))$ with negative $(L + k)$ -action; this yields that $k < c(L; G_0)$ and hence in particular $k < c(L; Q_0, Q_1)$, thus showing our claim. Consider a curve $u : [0, T] \rightarrow Q_0$ satisfying $\dot{u}(t) = -\mathcal{P}_0 w_{u(t)}$ for every $t \in [0, T]$ (observe that by assumption $\mathcal{P}_0 w_q \neq 0$ for every $q \in Q_0$); a straightforward computation shows that

$$S_k(u) = \int_0^T \left(k - \frac{1}{2} |\mathcal{P}_0 w_{u(t)}|^2 \right) dt \leq T \left(k - \min_{q \in Q_0} \frac{1}{2} |\mathcal{P}_0 w_q|^2 \right) \rightarrow -\infty$$

as $T \rightarrow +\infty$. Since Q_0 is compact and connected, for every $T > 0$ we can find a path $\gamma_T : [0, 1] \rightarrow Q_0$ connecting $u(T)$ with $u(0)$ and with $S_k(\gamma_T)$ uniformly bounded. It follows that, for T large enough, $\delta := \gamma_T \# u$ is a loop in Q_0 with negative $(L + k)$ -action.

In the general case consider the restriction $H : N^*Q_0 \rightarrow \mathbb{R}$; since H is Tonelli, there exists a smooth section $q_0 \mapsto \vartheta_{q_0}$ of the bundle $N^*Q_0 \rightarrow Q_0$. Consider now

$$u_{q_0} := \frac{\partial H}{\partial p}(q_0, \vartheta_{q_0}) \in T_{q_0} M.$$

Actually we have $u_{q_0} \in T_{q_0} Q_0$. Indeed if $p_0 \in N_{q_0}^* Q_0$ then

$$p_0(u_{q_0}) = \frac{\partial H}{\partial p}(q_0, \vartheta_{q_0})[p_0] = 0.$$

By definition we have

$$\min_{N^*Q_0} H = \min_{q_0 \in Q_0} E(q_0, u_{q_0}).$$

The assertion follows now, as above, integrating the vector field u on Q_0 long enough. \square

The discussion above actually also proves that the connected components of \mathcal{M}_Q correspond, with a slight abuse of notation, to the elements of $\pi_1(M, q_0)/\langle G_0, G_1 \rangle$.

Proposition 3.6. *The connected components of \mathcal{M}_Q correspond to the classes in $\pi_1(M, q_0)/\langle G_0, G_1 \rangle$.*

Proof. Observe that G_1 is not a subgroup of $\pi_1(M, q_0)$, but it can be naturally identified with a subgroup of it. Consider $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = q_0$ and $\gamma(1) = q_1$, the induced isomorphism

$$\psi : \pi_1(M, q_1) \rightarrow \pi_1(M, q_0), \quad \psi([\alpha]) := [\gamma^{-1} \# \alpha \# \gamma],$$

and the subgroup $\psi(G_1) < \pi_1(M, q_0)$. Obviously, $\psi(G_1)$ is independent of the choice of the path γ .

Notice furthermore that the connected components of \mathcal{M}_Q are in bijection with $\pi_0(\mathcal{M}_q)/\sim_{Q_0, Q_1}$, where \mathcal{M}_q is the space of paths connecting q_0 with q_1 and $[u] \sim_{Q_0, Q_1} [v]$ if and only if there exist $g_0 \in G_0$ and $g_1 \in G_1$ such that $[u] = [g_1 \# v \# g_0]$. The desired bijection is now given by

$$\pi_0(\mathcal{M}_q)/\sim_{Q_0, Q_1} \rightarrow \pi_1(M, q_0)/\langle G_0, G_1 \rangle, \quad [u] \mapsto [u \# \gamma]. \quad \square$$

Remark 3.7. If $Q_0 \cap Q_1$ is not connected, there might be more than one connected component of \mathcal{M}_Q containing constant paths as the following example shows. Consider \mathbb{T}^2 as the square $[0, 1]^2$ with identified sides and let Q_0 be a circle with center in $(\frac{1}{2}, \frac{1}{2})$ and radius $r_0 < \frac{1}{2}$. Let now Q_1 be another circle with center in $(0, \frac{1}{2})$ and radius $r_1 < \frac{1}{2}$ such that $r_0 + r_1 > \frac{1}{2}$ and denote with q_0, q_1, q_2, q_3 the four intersection points (see Figure 2). Clearly, the constant paths in q_0 and q_1 are contained in the same connected component of \mathcal{M}_Q ; the same holds for the constant paths in q_2 and q_3 . On the other hand, it is easy to see that the path

$$a : [0, 1] \rightarrow \mathbb{T}^2, \quad a(t) = ((q_0)_x + t, (q_0)_y)$$

is homotopic to the constant path in q_2 ; in particular, the constant paths in q_0 and in q_2 are in different connected components of \mathcal{M}_Q , since $[a] \neq 0$ in $\pi_1(M, q_0)/\langle G_0, G_1 \rangle = \pi_1(M, q_0)$.

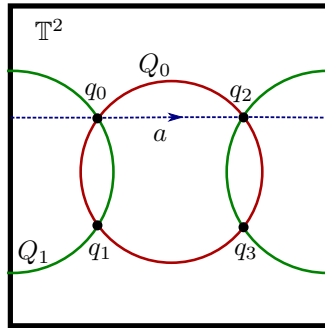


Figure 2. Constant paths may be contained in different connected components of \mathcal{M}_Q .

We show now that, for $k \geq c(L; Q_0, Q_1)$, \mathbb{A}_k is bounded from below on each connected component of \mathcal{M}_Q . The proof is analogous to the one of [1, Lemma 4.1], where the case of periodic orbits is treated and $c(L; Q_0, Q_1)$ is replaced by $c_u(L)$. Before proving this we need the following

Lemma 3.8. Let $G < \pi_1(M)$ be a normal subgroup containing G_0 and let $p_G : M_G \rightarrow M$ be the corresponding covering map, where $M_G := \tilde{M}/G$ is endowed with the metric obtained by lifting the given metric on M . Then there exists $Q_G \subseteq M_G$ such that $Q_G \cong Q_0$ and

$$p_G^{-1}(Q_0) = \bigcup_{\Gamma \in \pi_1(M)/G} \Gamma \cdot Q_G.$$

Proof. Denote with $\{U_j\}_{j \in J}$ the connected components of $p_G^{-1}(Q_0)$, so that $p_G^{-1}(Q_0) = \bigcup_{j \in J} U_j$. Observe that $p_G|_{U_j} : U_j \rightarrow Q_0$ is injective for every $j \in J$. Indeed, consider $p_0, p_1 \in U_j$ projecting to the same point $q_0 \in Q_0$ and pick any path $\alpha : [0, 1] \rightarrow U_j$ such that $\alpha(0) = p_0$ and $\alpha(1) = p_1$. Then $[p_G \circ \alpha] \in \iota_*(\pi_1(Q_0))$; this implies in particular that $[p_G \circ \alpha]$ is a non trivial Deck-transformation, which is clearly impossible since $G_0 \subseteq G$. It follows that $p_G|_{U_j} : U_j \rightarrow Q_0$ is a homeomorphism for every $j \in J$; in particular, U_j is compact for every $j \in J$.

Fix $Q_G := U_j$ for some $j \in J$. Clearly, for every $\Gamma \in \pi_1(M)/G$, $\Gamma \cdot Q_G$ is a connected component of $p_G^{-1}(Q_0)$. Conversely, let U_G be a connected component of $p_G^{-1}(Q_0)$. Consider $q_0 \in Q_0$ and its preimages $q \in Q_G$ and $u \in U_G$ under p_G . Since the cover is normal, the deck group $\pi_1(M)/G$ acts transitively on $p_G^{-1}(q_0)$; therefore, there exists $\Gamma \in \pi_1(M)/G$ such that $\Gamma(q) = u$ and hence $U_G \cap \Gamma \cdot Q_G \neq \emptyset$. It follows that $U_G = \Gamma \cdot Q_G$. \square

Lemma 3.9. For every $k \geq c(L; Q_0, Q_1)$ the free-time action functional \mathbb{A}_k is bounded from below on every connected component of \mathcal{M}_Q .

Proof. Consider $\sigma : [0, T] \rightarrow M$ in some connected component of \mathcal{M}_Q and

$$M_1 := \tilde{M}/\langle G_0, G_1 \rangle \xrightarrow{p} M, \tag{3.9}$$

where \tilde{M} is the universal cover. Denote by σ_1 the lift of σ to M_1 ; we lift the metric of M to M_1 and notice that, by Lemma 3.8, having fixed the connected component of \mathcal{M}_Q , we have that

$\text{dist}(\sigma_1(0), \sigma_1(T))$ is uniformly bounded. Therefore, there exists a path $\eta_1 : [0, 1] \rightarrow M_1$ which joins $\sigma_1(T)$ with $\sigma_1(0)$ and has uniformly bounded action

$$\tilde{S}_k(\eta_1) = \int_0^1 \left[L_1(\eta_1(t), \dot{\eta}_1(t)) + k \right] dt \leq C,$$

where L_1 denotes the lifted Lagrangian on M_1 . If $\eta := p \circ \eta_1$, then the juxtaposition $\sigma \# \eta \in \langle G_0, G_1 \rangle$ and, since by assumption $k \geq c(L; Q_0, Q_1)$, we get

$$0 \leq S_k(\sigma \# \eta) = \mathbb{A}_k(\sigma) + S_k(\eta) = \mathbb{A}_k(\sigma) + \tilde{S}_k(\eta_1) \leq \mathbb{A}_k(\sigma) + C$$

from which we deduce that $\mathbb{A}_k(\sigma) \geq -C$. \square

Corollary 3.10. *If $k > c(L; Q_0, Q_1)$, then every Palais-Smale sequence for \mathbb{A}_k in a connected component \mathcal{N} of \mathcal{M}_Q that does not contain constant paths admits a converging subsequence. The same holds if \mathcal{N} contains constant paths, provided that the Palais-Smale sequence is at level $c \neq 0$.*

Proof. Under the assumptions of the corollary we know by Lemmas 2.3 and 2.6 that the times T_h are bounded away from zero. Therefore, in virtue of Lemma 2.7 it is enough to show that the T_h 's are uniformly bounded from above. Since

$$\mathbb{A}_k(x, T) = \mathbb{A}_{c(L; Q_0, Q_1)}(x, T) + (k - c(L; Q_0, Q_1))T$$

for any $(x, T) \in \mathcal{M}_Q$, the period

$$T_h = \frac{1}{k - c(L; Q_0, Q_1)} \left[\mathbb{A}_k(x_h, T_h) - \mathbb{A}_{c(L; Q_0, Q_1)}(x_h, T_h) \right]$$

is clearly uniformly bounded from above, being \mathbb{A}_k bounded on the Palais-Smale sequence and being $\mathbb{A}_{c(L; Q_0, Q_1)}(x_h, T_h)$ bounded from below by Lemma 3.9. \square

When looking for connecting orbits in case Q_0, Q_1 intersect, there is another relevant energy value which we now define. In the next section we will namely use Corollary 3.10 to construct orbits satisfying the conormal boundary conditions as action minimizers. However, when minimizing on a connected component containing constant paths we need to ensure that the infimum is negative (observe that such an infimum cannot be positive). This is not always the case as the example in the introduction shows. Thus, let \mathcal{N} be a connected component of \mathcal{M}_Q containing constant paths and define the energy value

$$k_{\mathcal{N}}(L) := \inf \left\{ k \in \mathbb{R} \mid \mathbb{A}_k(\gamma) \geq 0, \forall \gamma \in \mathcal{N} \right\}. \quad (3.10)$$

By definition we readily see that $c(L; Q_0, Q_1) \leq k_{\mathcal{N}}(L)$. In the next section we show that in the interval $(c(L; Q_0, Q_1), k_{\mathcal{N}}(L))$ we find Euler-Lagrange orbits in \mathcal{N} satisfying the conormal boundary conditions by minimizing \mathbb{A}_k on \mathcal{N} . Notice that the considered interval might be empty but in general it is not; see Section 6 for an example. Existence results above $k_{\mathcal{N}}(L)$ are in general achievable only under additional assumptions (c.f. Theorem 1).

Another “natural” energy value is given by

$$k_0(L) := \inf \left\{ k \in \mathbb{R} \mid \mathbb{A}_k(\gamma) \geq 0, \forall \gamma \in \mathcal{M}_Q \right\}.$$

It is interesting to study the relation between $k_0(L)$ and the critical value $c(L; Q_0, Q_1)$ and, more generally, the other critical values we introduced in this section; this will also give us an estimate on how much the various critical values can differ.

Clearly $c(L; Q_0, Q_1) \leq k_0(L)$. We claim that actually $c(L) \leq k_0(L)$. Thus, consider $k < c(L)$; by definition there exists a loop δ such that $S_k(\delta) < 0$. It is now easy to construct a path from Q_0 to Q_1 with negative action: pick any path η from a point $q_0 \in Q_0$ to the base point $\delta(0)$, then wind n -times around δ and finally join $\delta(0)$ with a point $q_1 \in Q_1$ by a path μ . If n is large enough then

$$\mathbb{A}_k(\mu \# \delta^n \# \eta) = S_k(\mu) + nS_k(\delta) + S_k(\eta) < 0,$$

which implies $k < k_0(L)$ and the claim follows. Therefore, we have

$$e_0(L) \leq c_u(L) \leq c(L; Q_0, Q_1) \leq c(L) \leq k_0(L),$$

where the second and third inequalities follow from Lemma 3.1. It is easy to see that in general there is no relation between $c_0(L)$ and $c(L; Q_0, Q_1)$.

In order to estimate how much the various Mañé critical values can differ, one can measure the difference $k_0(L) - e_0(L)$. Thus, consider the smooth one-form

$$\vartheta(q)[v] := d_v L(q, 0)[v];$$

by taking a Taylor expansion and by using (2.2), we get that

$$\begin{aligned} L(q, v) &= L(q, 0) + d_v L(q, 0)[v] + \frac{1}{2} d_{vv} L(q, 0)[v, v] \\ &\geq -E(q, 0) + \vartheta(q)[v] + a|v|^2, \end{aligned}$$

where $s \in [0, 1]$ is a suitable number. If we set $\gamma(t) := x(t/T)$, then we obtain

$$\begin{aligned} \mathbb{A}_k(x, T) &= \mathbb{A}_k(\gamma) \geq \int_0^T \left[-E(\gamma(t), 0) + \vartheta(\gamma(t))[\dot{\gamma}(t)] + a|\dot{\gamma}(t)|^2 + k \right] dt \\ &= \int_0^T \left[k - E(\gamma(t), 0) \right] dt + \int_0^T \gamma^* \vartheta + a \int_0^T |\dot{\gamma}(t)|^2 dt \\ &\geq \left[k - e_0(L) \right] T + \frac{a}{T} l(\gamma)^2 - \|\vartheta\|_\infty l(\gamma). \end{aligned}$$

For $k > e_0(L)$ and T fixed, the latter expression is a parabola in $l(\gamma)$ with minimum

$$(k - e_0(L))T - \frac{\|\vartheta\|_\infty^2}{4a} T = \left(k - e_0(L) - \frac{\|\vartheta\|_\infty^2}{4a} \right) T.$$

In particular, if

$$k > e_0(L) + \frac{\|\vartheta\|_\infty^2}{4a},$$

then $\mathbb{A}_k(\gamma) \geq 0$ for any path γ connecting Q_0 with Q_1 and this implies, by the definition of $k_0(L)$, that $k > k_0(L)$. Therefore we get

$$k_0(L) \leq e_0(L) + \frac{\|\vartheta\|_\infty^2}{4a}.$$

We sum up the discussion above with the following

Proposition 3.11. *Let $L : TM \rightarrow \mathbb{R}$ be a Tonelli Lagrangian, $Q_0, Q_1 \subseteq M$ closed submanifolds. Then the following chain of inequalities holds*

$$e_0(L) \leq c_u(L) \leq c(L; Q_0, Q_1) \leq c(L) \leq k_0(L) \leq e_0(L) + \frac{\|\vartheta\|_\infty^2}{4a}. \quad (3.11)$$

In particular, if $\vartheta \equiv 0$, that is if L is a mechanic Lagrangian as in (1.6), we retrieve

$$e_0(L) = c_u(L) = c(L; Q_0, Q_1) = c(L) = k_0(L).$$

4. EXISTENCE RESULTS FOR HIGH ENERGIES

In this section, building on the analytical backgrounds introduced in the previous ones, we prove Theorem 1. However, before proving the Theorem we need some preliminaries.

Thus, suppose that $Q_0, Q_1 \subseteq M$ are two closed connected intersecting submanifolds and let \mathcal{N} be the connected component of \mathcal{M}_Q containing the constant paths. Assume in addition that $Q_0 \cap Q_1$ is connected. We say that \mathcal{N} has the *retraction property* if there exists a neighborhood \mathcal{U} of $Q_0 \cap Q_1$ in \mathcal{N} such that $Q_0 \cap Q_1$ is a strong deformation retract of \mathcal{U} .

Remark 4.1. \mathcal{N} has the retraction property, for instance, if $Q_0 \cap Q_1 = \{q\}$ or if there exists $\epsilon > 0$ small enough such that there are no geodesics with length $0 < \ell < \epsilon$ from Q_0 to Q_1 and hitting Q_0 and Q_1 orthogonally. In the first case indeed, if $B_i \subseteq Q_i$ is a contractible neighborhood of q in Q_i , for $i = 0, 1$, then \mathcal{U} retracts on the image of the map $B_0 \times B_1 \hookrightarrow \mathcal{U}$ which associates to every (q_0, q_1) the shortest geodesic connecting q_0 with q_1 . Since $B_0 \times B_1$ retracts on (q, q) , the conclusion follows.

If the second property holds then we can use the negative gradient flow of the kinetic energy

$$e : H_Q^1([0, 1], M) \rightarrow \mathbb{R}, \quad e(x) = \int_0^1 |x'(s)|^2 ds$$

to deform the set $\{(x, T) \in \mathcal{N} \mid e(x) < \epsilon^2\}$ into the set of constant paths. Notice that this latter property holds for $Q_0 = Q_1$ and for generic choice of Q_0 and Q_1 .

Proof of Theorem 1. Suppose \mathcal{N}' is a connected component of \mathcal{M}_Q not containing constant paths. Lemma 2.3 implies that the sublevels of \mathbb{A}_k in \mathcal{N}'

$$\{(x, T) \in \mathcal{N}' \mid \mathbb{A}_k(x, T) \leq c\}$$

are complete. Moreover, Corollary 3.10 implies that \mathbb{A}_k satisfies the Palais-Smale condition on \mathcal{N}' for every $k > c(L; Q_0, Q_1)$. We may then conclude that \mathbb{A}_k has a global minimizer on \mathcal{N}' by taking a minimizing sequence for \mathbb{A}_k as Palais-Smale sequence.

We prove now statement 2, (a). Thus, let \mathcal{N} be the connected component of \mathcal{M}_Q containing the constant paths. Consider first $k \in (c(L; Q_0, Q_1), k_{\mathcal{N}}(L))$; since $c := \inf \mathbb{A}_k < 0$, the sublevel sets of \mathbb{A}_k in \mathcal{N}

$$\{(x, T) \in \mathcal{N} \mid \mathbb{A}_k(x, T) \leq c + \epsilon\}$$

are complete for every $\epsilon > 0$ small by Corollary 2.4. Moreover, Lemma 2.6 implies that all the Palais-Smale sequences at level c have T_h 's bounded away from zero and hence \mathbb{A}_k satisfies the Palais-Smale condition at level c by Lemma 3.9. We now retrieve the existence of a global minimizer for \mathbb{A}_k in \mathcal{N} exactly as above.

Suppose now that $Q_0 \cap Q_1$ is connected, \mathcal{N} has the retraction property, and there exists $l \geq 1$ such that $\pi_l(\mathcal{N}, Q_0 \cap Q_1) \neq 0$. Fix $k > k_{\mathcal{N}}(L)$; in this case, we retrieve the desired Euler-Lagrange orbit using a minimax argument analogous to that used by Lusternik and Fet [17] in their proof of the existence of one closed geodesic on a simply connected manifold (see also [1] or [11] for an application to the existence of periodic Euler-Lagrange orbits; in that case $k_{\mathcal{N}}(L)$ is replaced by $c_u(L)$). By assumption there exists a non-trivial element $\mathcal{H} \in \pi_l(\mathcal{N}, Q_0 \cap Q_1)$ and therefore we can consider the minimax value

$$c := \inf_{h \in \mathcal{H}} \max_{\zeta \in D^l} \mathbb{A}_k(h(\zeta)).$$

Let us show that $c > 0$; since \mathcal{H} is non-trivial, there exists a positive number λ such that for every map $h = (x, T) : (D^l, S^{l-1}) \rightarrow (\mathcal{N}, Q_0 \cap Q_1)$ belonging to the class \mathcal{H} there holds

$$\max_{\zeta \in D^l} l(x(\zeta)) \geq \lambda,$$

where as usual $l(x(\zeta))$ denotes the length of the path $x(\zeta)$. This follows simply from the fact that $\inf_{\gamma \in \partial \mathcal{U}} l(\gamma) > 0$. If $(x, T) \in \mathcal{N}$ has length $l(x) \geq \lambda$, then (2.1) implies that

$$\begin{aligned} \mathbb{A}_k(x, T) &= T \int_0^1 \left[L\left(x(s), \frac{x'(s)}{T}\right) + k \right] ds \\ &\geq \frac{a}{T} \int_0^1 |x'(s)|^2 ds + T(k - b) \\ &\geq \frac{a}{T} l(x)^2 + T(k - b) \\ &\geq \frac{a}{T} \lambda^2 + T(k - b). \end{aligned}$$

Since $\lambda > 0$, the above inequality implies that if $(x, T) \in \mathcal{N}$ has length $l(x) \geq \lambda$ and action $\mathbb{A}_k(x, T) \leq c + 1$ then

$$c + 1 \geq \frac{a}{T} \lambda^2 + T(k - b)$$

and hence $T \geq T_0$ for some $T_0 > 0$, because the quantity on the righthand-side goes to infinity as $T \rightarrow 0$. Now let $h \in \mathcal{H}$ be such that

$$\max_{\zeta \in D^l} \mathbb{A}_k(h(\zeta)) \leq c + 1;$$

then by the above considerations there exists $(x, T) \in h(D^l)$ with $T \geq T_0$ and

$$\mathbb{A}_k(x, T) = \mathbb{A}_{k_{\mathcal{N}}(L)}(x, T) + (k - k_{\mathcal{N}}(L))T \geq (k - k_{\mathcal{N}}(L))T_0 > 0.$$

The argument above shows that the minimax value c is strictly positive. The celebrated minimax theorem (see for instance [1, Theorem 1.8 and Remark 1.11] or [5, Theorem 2.5.3]), combined with Lemma 2.5, ensures the existence of a Palais-Smale sequence at level c . Since $c > 0$ we also get from Lemma 2.6 that the T_h 's are bounded away from zero, so that by Corollary 3.10 the Palais-Smale sequence has a limit point in \mathcal{N} , which gives us the required connecting orbit, thus proving 2, (b).

Suppose now that $Q_0 \cap Q_1$ is not connected and at least one connected component Ω of $Q_0 \cap Q_1$ is isolated; in this case it is easy to see that there are always non-trivial classes in $\pi_1(\mathcal{N}, Q_0 \cap Q_1)$. Indeed, any continuous map $u : (D^1, S^0) \rightarrow (\mathcal{N}, Q_0 \cap Q_1)$ such that $u(0) \in \Omega$ and $u(1) \in \Omega'$ for some $\Omega' \neq \Omega$ represents a non-trivial class in $\pi_1(\mathcal{N}, Q_0 \cap Q_1)$. Moreover, for every such a continuous map u we have

$$\max_{s \in [0,1]} l(u(s)) \geq \inf_{\partial \mathcal{U}} l > 0,$$

where \mathcal{U} is an open set such that $\Omega \subseteq \mathcal{U}$ and $\Omega' \cap \bar{\mathcal{U}} = \emptyset$. The proof of 2, (c) follows now repeating the argument used to prove 2, (b). \square

Proof of Corollary 1. The first statement and the first part of the second one follow trivially from the corresponding statements of Theorem 1. The second part of the second statement follows from Theorem 1 and the fact that $\pi_l(\mathcal{N}, Q_0) \cong \pi_{l+1}(M, Q_0)$, for every $l \geq 0$.

Thus it remains to show that for every $k > c(L; Q_0)$, $k \neq k_{\mathcal{N}}(L)$, there always exists an Arnold chord if $Q_0 \neq M$. By connectedness of the spaces we have $\pi_0(M, Q_0) = \{0\}$. Moreover, if \mathcal{N} is the only connected component of \mathcal{M}_Q , then we also have $\pi_1(M, Q_0) = \{0\}$. Therefore, if \mathcal{N} is the only connected component of \mathcal{M}_Q and $\pi_l(\mathcal{N}, Q_0) = \{0\}$ for every $l \geq 1$, then $\pi_l(M, Q_0) = \{0\}$ for every $l \geq 0$. But then the Hurewicz theorem (cf. [19, Theorem 4.37]) would yield $H_l(M, Q_0) = 0$ for every $l \in \mathbb{N}$ and, hence, $H_l(M) \cong H_l(Q_0)$, which is a contradiction for $l = \dim M$. \square

5. EXISTENCE RESULTS FOR LOW ENERGIES

In this section we study the existence of Euler-Lagrange orbits satisfying the conormal boundary conditions for subcritical energies $k < c(L; Q_0, Q_1)$. As already explained in the introduction, this problem is harder than the corresponding one for supercritical energies.

Throughout this section we assume that the submanifolds Q_0 and Q_1 intersect; we will get back to the case $Q_0 \cap Q_1 = \emptyset$ in Section 6 showing that, under this assumption, the first statement of Theorem 1 is optimal.

We start by considering the following particular case, which should help to understand the general situation later on. Let Q_0 and Q_1 be two closed connected submanifolds which intersect in one point, say p . We show that, under the assumption (5.3) on the Lagrangian L , for every $k \in (E(p, 0), c(L; Q_0, Q_1))$ the action functional \mathbb{A}_k exhibits a mountain-pass geometry on the connected component \mathcal{N} of \mathcal{M}_Q that contains the constant paths.

Since for any $k \in (E(p, 0), c(L; Q_0, Q_1))$ the free-time action functional \mathbb{A}_k is unbounded from below, it makes sense to define the following class of paths in \mathcal{N}

$$\Gamma := \left\{ u : [0, 1] \rightarrow \mathcal{N} \mid u(0) = (p, T), T \leq T_0, \mathbb{A}_k(u(1)) < 0 \right\}, \quad (5.1)$$

where $T_0 > 0$ is chosen so small that the class Γ is invariant under (a suitable truncation of) the negative gradient flow of \mathbb{A}_k . The existence of T_0 will be shown in Remark 5.1. Define now

$$\vartheta_q(\cdot) := d_v L(q, 0)[\cdot], \quad \forall q \in M, \quad (5.2)$$

and assume that there exists an open neighborhood \mathcal{U} of p such that

$$\vartheta_q \equiv 0, \quad \forall q \in \mathcal{U}. \quad (5.3)$$

Without loss of generality we may suppose that $\mathcal{U} = B_r$ is an open ball with radius r around p . Under the assumption (5.3) we show the desired mountain-pass geometry for the action functional \mathbb{A}_k . Namely, we prove that there is $\alpha > 0$ such that

$$\max_{s \in [0,1]} \mathbb{A}_k(u(s)) \geq \alpha, \quad \forall u \in \Gamma.$$

Here is the scheme of the proof: we first show that if the length of a path γ connecting Q_0 and Q_1 is sufficiently small then the action of γ needs to be non-negative. Therefore, for every element $u \in \Gamma$ there must be an $s \in [0, 1]$ such that $l(u(s)) = \epsilon$ for a suitable $\epsilon > 0$. Now we get the assertion showing that every path with length ϵ has \mathbb{A}_k -action bounded away from zero by a positive constant.

Since Q_0 and Q_1 intersect only in p , for every $\delta > 0$ there exists $\lambda_\delta > 0$ such that

$$d(Q_0 \setminus B_\delta, Q_1 \setminus B_\delta) \geq \lambda_\delta,$$

where B_δ denotes the ball with radius δ around p . In other words, every path connecting Q_0 to Q_1 with starting and ending point outside B_δ has length larger than λ_δ . It is clear now that, if $\epsilon > 0$ is sufficiently small, then every path γ connecting Q_0 to Q_1 with length $l(\gamma) \leq \epsilon$ is entirely contained in $\mathcal{U} = B_r$. Indeed, fix $\delta < r$. If $\epsilon < \min\{r - \delta, \lambda_\delta\}$, then at least one between the starting and ending point of γ is contained in B_δ , say $\gamma(0) \in B_\delta$, and

$$d(\gamma(t), p) < d(\gamma(t), \gamma(0)) + d(\gamma(0), p) < \epsilon + \delta < r, \quad \forall t. \quad (5.4)$$

A Taylor expansion together with the bound (2.2) implies

$$\begin{aligned} L(q, v) &= L(q, 0) + d_v L(q, 0)[v] + \frac{1}{2} d_{vv} L(q, 0)[v, v] \\ &\geq -E(q, 0) + \vartheta_q(v) + a|v|^2. \end{aligned} \quad (5.5)$$

Let now $k > E(p, 0)$. Up to choosing a smaller neighborhood \mathcal{U} of p (thus, a smaller ϵ), the continuity of the energy implies that

$$k > \sup_{q \in \mathcal{U}} E(q, 0).$$

Using (5.3), (5.4) and (5.5) we now compute for every $\gamma = (x, T)$ with length $l(x) \leq \epsilon$

$$\begin{aligned} \mathbb{A}_k(x, T) &\geq \int_0^T \left[-E(\gamma(t), 0) + \vartheta_{\gamma(t)}(\gamma'(t)) + a|\gamma'(t)|^2 + k \right] dt \\ &\geq T(k - E(p, 0)) + \frac{a}{T} l(x)^2 \end{aligned}$$

which is a non-negative quantity. It follows that for every $u \in \Gamma$ there is $s \in [0, 1]$ such that $l(u(s)) = \epsilon$; for such s we obtain

$$\begin{aligned} \mathbb{A}_k(u(s)) &\geq T(k - e_0(L)) + \frac{a}{T} \epsilon^2 \\ &\geq 2\epsilon \sqrt{a(k - e_0(L))} \\ &=: \alpha \end{aligned}$$

as we wished to prove.

Remark 5.1. Since $\mathbb{A}_k(p, T) = T(k - E(p, 0)) > 0$ goes to zero as $T \rightarrow 0$, we can choose T_0 in the definition of Γ such that $\mathbb{A}_k(p, T_0) \leq \alpha/4$. In this way the class Γ becomes invariant under the negative gradient flow of \mathbb{A}_k truncated below level $\alpha/2$.

We are now ready to deal with the general case. Let $Q_0, Q_1 \subseteq M$ be closed and connected submanifolds with non-empty intersection. Let $\Omega \subseteq Q_0 \cap Q_1$ be an isolated connected component of $Q_0 \cap Q_1$ and denote with \mathcal{N} the connected component of \mathcal{M}_Q containing the constant paths. Define

$$k_\Omega := \min \left\{ c(L; Q_0, Q_1), \max_{q \in \Omega} E(q, 0) + \max_{q \in \Omega} \frac{|\vartheta_q|^2}{4a} \right\}, \quad (5.6)$$

where ϑ_q is as in (5.2), $|\cdot|$ is the dual norm on T^*M induced by the Riemannian metric on M and $a > 0$ is such that (2.2) is satisfied. Now set the minimax class

$$\Gamma := \left\{ u = (x, T) : [0, 1] \rightarrow \mathcal{N} \mid x(0) = p \in \Omega, T(0) \leq T_0, \mathbb{A}_k(u(1)) < 0 \right\}, \quad (5.7)$$

where as above T_0 will be chosen such that $\mathbb{A}_k(p, T_0) \leq \alpha/4$ for every $p \in \Omega$, where $\alpha > 0$ will be the constant given by Lemma 5.2. In this way Γ will become invariant under the negative gradient flow of \mathbb{A}_k truncated below level $\alpha/2$.

Lemma 5.2 states that, for every $k \in (k_\Omega, c(L; Q_0, Q_1))$, the action functional \mathbb{A}_k has a mountain-pass geometry on \mathcal{N} , where the two valleys are represented by the constant paths (in Ω) and by

the paths with negative action. Notice however, that the considered interval could be empty; this happens, for instance, when

$$\max_{q \in \Omega} E(q, 0) = e_0(L), \quad \max_{q \in \Omega} \frac{|\vartheta_q|^2}{4a} = \frac{\|\vartheta\|_\infty^2}{4a},$$

as the chain of inequalities (3.11) shows. However, this is not always the case as we will show in the counterexample section. Observe also that, when $Q_0 \cap Q_1 = \{p\}$ with $\vartheta_p = 0$, the energy value $k_{Q_0 \cap Q_1}$ reduces to the above considered $E(p, 0)$.

Lemma 5.2. *Let $Q_0, Q_1 \subseteq M$ be two closed connected submanifolds with non-empty intersection, let Ω be an isolated connected component of $Q_0 \cap Q_1$ and let k_Ω be as defined in (5.6). For every $k \in (k_\Omega, c(L; Q_0, Q_1))$ there exists $\alpha > 0$ such that*

$$\inf_{u \in \Gamma} \max_{s \in [0, 1]} \mathbb{A}_k(u(s)) \geq \alpha.$$

Proof. The proof follows from the one in the particular case treated above with minor adjustments. Consider a neighborhood \mathcal{U} of Ω such that

$$k > \sup_{q \in \mathcal{U}} E(q, 0) + \sup_{q \in \mathcal{U}} \frac{|\vartheta_q|^2}{4a}. \quad (5.8)$$

As in the particular case one shows now that, if $\epsilon > 0$ is sufficiently small, then every path joining Q_0 to Q_1 with length less than or equal to ϵ and contained in the connected component of $\{(x, T) \in \mathcal{N} \mid l(x) \leq \epsilon\}$ containing Ω has image contained in \mathcal{U} (here is where we need the assumption Ω isolated). Pick now such an ϵ ; using (5.5) we compute for every $\gamma = (x, T)$ with $l(x) \leq \epsilon$

$$\begin{aligned} \mathbb{A}_k(x, T) &\geq \int_0^T \left[-E(\gamma(t), 0) + \vartheta_{\gamma(t)}(\gamma'(t)) + a|\gamma'(t)|^2 + k \right] dt \\ &\geq \left(k - \sup_{q \in \mathcal{U}} E(q, 0) \right) T + \frac{a}{T} l(x)^2 + \int_0^T \vartheta_{\gamma(t)}(\gamma'(t)) dt \\ &\geq \left(k - \sup_{q \in \mathcal{U}} E(q, 0) \right) T + \frac{a}{T} l(x)^2 - \left(\sup_{q \in \mathcal{U}} |\vartheta_q| \right) l(x). \end{aligned}$$

To ease the notation let us define

$$c_E := \sup_{q \in \mathcal{U}} E(q, 0), \quad c_\vartheta := \sup_{q \in \mathcal{U}} |\vartheta_q|$$

and consider the function of two variables

$$f : (0, +\infty) \times [0, \epsilon] \rightarrow \mathbb{R}, \quad f(T, l) := (k - c_E)T + \frac{a}{T} l^2 - c_\vartheta l.$$

For every l fixed the function f has minimum

$$\left(2\sqrt{a(k - c_E)} - c_\vartheta \right) l$$

and this quantity is positive if and only if (5.8) is satisfied. Now, arguing as above, we get that for every $u \in \Gamma$ there exists $s \in [0, 1]$ such that $l(u(s)) = \epsilon$. For this s we readily have

$$\mathbb{A}_k(u(s)) \geq \left(2\sqrt{a(k - c_E)} - c_\vartheta \right) \epsilon =: \alpha > 0,$$

exactly as we wished to prove. □

Remark 5.3. *When Ω consists of more than one point one would be tempted to replace in the definition of k_Ω the maximum of the energy on Ω with the corresponding minimum, hence defining*

$$k_\Omega^- := \min \left\{ c(L; Q_0, Q_1), \min_{q \in \Omega} E(q, 0) + \max_{q \in \Omega} \frac{|\vartheta_q|^2}{4a} \right\}, \quad (5.9)$$

and show that the conclusion of Lemma 5.2 holds even considering the a priori larger interval $(k_\Omega^-, c(L; Q_0, Q_1))$. This is however not the case, since under these assumptions there are constant paths with negative \mathbb{A}_k -action. However, it seems reasonable to us that an argument analogous to the one in [1], where the case of periodic orbits is considered and k_Ω^-, k_Ω are replaced by $\min E, e_0(L)$

respectively, should go through in this setting, at least under some mild additional assumptions (such as Ω being a CW-complex). Namely, in the energy range (k_Ω^-, k_Ω) , instead of the class Γ , one should consider the class of deformations $u = (x, T) : [0, 1] \times \Omega \rightarrow \mathcal{N}$ of the space of constant paths into the space of paths with negative \mathbb{A}_k -action

$$\Gamma_\Omega := \left\{ u = (x, T) \mid x(0, q) = q, \mathbb{A}_k(u(1, q)) < 0, \forall q \in \Omega \right\}.$$

However, it is a priori not clear why the class Γ_Ω should be non-empty. Indeed, in order to show that the corresponding class in the periodic setting is non-empty, besides the CW-complex structure of M one has to use the iteration of loops and Bangert's trick of pulling one loop at a time [8] (we refer again to [1] and references therein for the details), which do not immediately generalize to our setting. This will be subject of future research.

If Ω is not isolated then the proof of Lemma 5.2 might fail. Nevertheless, we can define the energy value k_Ω in a suitable fashion so that the conclusion of Lemma 5.2 still holds. Thus, we say that a collection ν of connected components of $Q_0 \cap Q_1$ is an *isolating family* for Ω if $\Omega \in \nu$ and there exists $\epsilon > 0$ such that $B_\epsilon(\nu) \cap \Omega' = \emptyset$ for every connected component Ω' of $Q_0 \cap Q_1$ that is not contained in ν . Notice that the union of all connected components of $Q_0 \cap Q_1$ is an isolating family for every connected component (in particular, isolating families always exist) and that $\nu = \{\Omega\}$ is an isolating family if Ω is isolated.

With slight abuse of notation we denote with ν both the isolating family and the union of all sets in the isolating family. For every isolating family ν for Ω we define

$$k_\nu := \min \left\{ c(L; Q_0, Q_1), \max_{q \in \nu} E(q, 0) + \max_{q \in \nu} \frac{|\vartheta_q|^2}{4a} \right\}.$$

It is easy to see now that the argument in the proof of Lemma 5.2 goes through replacing k_Ω with k_ν . Since this holds for every isolating family we can define

$$k_\Omega := \inf \left\{ k_\nu \mid \nu \text{ isolating family for } \Omega \right\}. \quad (5.10)$$

It is easy to see that this definition of k_Ω coincides with the one given in (5.6) if Ω is isolated. Indeed, there clearly holds $k_{\nu_1} \leq k_{\nu_2}$ if $\nu_1 \subseteq \nu_2$ (meaning that every connected component of $Q_0 \cap Q_1$ contained in ν_1 is also contained in ν_2), and $\{\Omega\}$ is the smallest isolating family for Ω if Ω is isolated.

The next lemma follows now directly from Lemma 5.2, keeping in mind the new definition of k_Ω .

Lemma 5.4. *Let $Q_0, Q_1 \subseteq M$ be two closed connected submanifolds with non-empty intersection, let Ω be a connected component of $Q_0 \cap Q_1$ and let k_Ω be as defined in (5.10). For every $k \in (k_\Omega, c(L; Q_0, Q_1))$ there exists $\alpha > 0$ such that*

$$\inf_{u \in \Gamma} \max_{s \in [0, 1]} \mathbb{A}_k(u(s)) \geq \alpha.$$

We can now define the minimax function

$$c_\Omega : (k_\Omega, c(L; Q_0, Q_1)) \longrightarrow \mathbb{R}, \quad c_\Omega(k) := \inf_{u \in \Gamma} \max_{[0, 1]} \mathbb{A}_k \circ u. \quad (5.11)$$

Lemma 5.2 above implies that $c_\Omega(k) > 0$ for all k ; furthermore, the monotonicity of \mathbb{A}_k in k implies that the minimax function $c_\Omega(\cdot)$ is monotonically increasing and hence almost everywhere differentiable. In Lemma 5.5 we prove the existence of bounded Palais-Smale sequences for every value of the parameter k at which the minimax functions $c_\Omega(\cdot)$ is differentiable, thus overcoming the lack of the Palais-Smale condition for \mathbb{A}_k for subcritical energies. The proof is analogous to the one in the periodic case (see [11] and [1] for further details) and is based on the celebrated *Struwe monotonicity argument* (cf. [27]).

Lemma 5.5. *Suppose that \bar{k} is a point of differentiability for the minimax function $c_\Omega(\cdot)$ in (5.11). Then $\mathbb{A}_{\bar{k}}$ admits a bounded Palais-Smale sequence at level $c_\Omega(\bar{k})$.*

Proof. Since \bar{k} is a point of differentiability for $c_\Omega(\cdot)$ we have

$$|c_\Omega(k) - c_\Omega(\bar{k})| \leq M|k - \bar{k}| \quad (5.12)$$

for all k sufficiently close to \bar{k} , where $M > 0$ is a suitable constant. Let $\{k_h\}$ be a strictly decreasing sequence which converges to \bar{k} and set $\epsilon_h := k_h - \bar{k} \downarrow 0$. For every $h \in \mathbb{N}$ choose $u_h \in \Gamma$ (or $\Gamma_{Q_0 \cap Q_1}$) such that

$$\max_{u_h} \mathbb{A}_{k_h} \leq c_\Omega(k_h) + \epsilon_h.$$

Up to ignoring a finite numbers of k_h 's we may suppose that equation (5.12) is satisfied by every k_h . If $z = (x, T) \in u_h$ is such that $\mathbb{A}_{\bar{k}}(z) > c_\Omega(\bar{k}) - \epsilon_h$, then

$$T = \frac{\mathbb{A}_{k_h}(z) - \mathbb{A}_{\bar{k}}(z)}{k_h - \bar{k}} \leq \frac{c_\Omega(k_h) + \epsilon_h - c_\Omega(\bar{k}) + \epsilon_h}{\epsilon_h} \leq M + 2.$$

Moreover,

$$\mathbb{A}_{\bar{k}}(z) \leq \mathbb{A}_{k_h}(z) \leq c_\Omega(k_h) + \epsilon_h \leq c_\Omega(\bar{k}) + (M + 1)\epsilon_h$$

and hence

$$u_h \subseteq \mathcal{A}_h \cup \left\{ \mathbb{A}_{\bar{k}} \leq c_\Omega(\bar{k}) - \epsilon_h \right\},$$

where

$$\mathcal{A}_h = \left\{ (x, T) \in \mathcal{N} \mid T \leq M + 2, \mathbb{A}_{\bar{k}}(x, T) \leq c_\Omega(\bar{k}) + (M + 1)\epsilon_h \right\}.$$

Observe that, if $(x, T) \in \mathcal{A}_h$, then by (2.1) we have

$$\mathbb{A}_{\bar{k}}(x, T) \geq \frac{a}{M + 2} \|x'\|_2^2 - (M + 2)|b - \bar{k}|$$

and hence

$$\|x'\|_2^2 \leq \frac{M + 2}{a} \left(c_\Omega(\bar{k}) + (M + 1)\epsilon_h + (M + 2)|b - \bar{k}| \right),$$

which shows that \mathcal{A}_h is bounded in \mathcal{N} , uniformly in h . Let Φ be the flow of the vector field obtained by multiplying $-\nabla \mathbb{A}_{\bar{k}}$ by a suitable non-negative function, whose role is to make the vector field bounded on \mathcal{N} and vanishing on the sublevel $\{\mathbb{A}_{\bar{k}} \leq c_\Omega(\bar{k})/2\}$, while keeping the uniform decrease condition

$$\frac{d}{d\sigma} \mathbb{A}_{\bar{k}}(\Phi_\sigma(z)) \leq -\frac{1}{2} \min \left\{ \|d\mathbb{A}_{\bar{k}}(\Phi_\sigma(z))\|^2, 1 \right\}, \quad \text{if } \mathbb{A}_{\bar{k}}(\Phi_\sigma(z)) \geq \frac{c_\Omega(\bar{k})}{2}. \quad (5.13)$$

Lemma 2.5 implies that Φ is well-defined on $[0, +\infty) \times \mathcal{N}$ and that Γ is positively invariant with respect to Φ . Since Φ maps bounded sets into bounded sets,

$$\Phi([0, 1] \times u_h) \subseteq \mathcal{B}_h \cup \left\{ \mathbb{A}_{\bar{k}} \leq c_\Omega(\bar{k}) - \epsilon_h \right\} \quad (5.14)$$

for some uniformly bounded set

$$\mathcal{B}_h \subseteq \left\{ \mathbb{A}_{\bar{k}} \leq c_\Omega(\bar{k}) + (M + 1)\epsilon_h \right\}. \quad (5.15)$$

We claim that there exists a sequence $\{z_h\} \subseteq \mathcal{N}$ with

$$z_h \in \mathcal{B}_h \cap \left\{ \mathbb{A}_{\bar{k}} \geq c_\Omega(\bar{k}) - \epsilon_h \right\}$$

and $\|d\mathbb{A}_{\bar{k}}(z_h)\|$ infinitesimal. Such a sequence is clearly a bounded Palais-Smale sequence at level $c_\Omega(\bar{k})$. Assume by contradiction that there exists $\delta \in (0, 1)$ such that

$$\|d\mathbb{A}_{\bar{k}}\| \geq \delta, \quad \text{on } \mathcal{B}_h \cap \left\{ \mathbb{A}_{\bar{k}} \geq c_\Omega(\bar{k}) - \epsilon_h \right\}$$

for every h large enough. Together with (5.13), (5.14) and (5.15), this implies that, for h large enough, for any $z \in u_h$ such that

$$\Phi([0, 1] \times \{z\}) \subseteq \left\{ \mathbb{A}_{\bar{k}} \geq c_\Omega(\bar{k}) - \epsilon_h \right\}$$

there holds

$$\mathbb{A}_{\bar{k}}(\Phi_1(z)) \leq \mathbb{A}_{\bar{k}}(z) - \frac{1}{2}\delta^2 \leq c_\Omega(\bar{k}) + (M + 1)\epsilon_h - \frac{1}{2}\delta^2.$$

It follows that

$$\max_{\Phi_1(u_h)} \mathbb{A}_{\bar{k}} \leq c_\Omega(\bar{k}) - \epsilon_h$$

for h large enough. Since $\Phi_1(u_h) \in \Gamma$, this contradicts the definition of $c(\bar{k})$. □

Proof of Theorem 3. Follows combining Lemma 5.5 above with Lemma 2.7 and with the fact that a monotonically increasing function is differentiable almost everywhere. \square

Remark 5.6. *The minimax functions c_Ω do not provide in general different critical points of \mathbb{A}_k . The only convenience to pick one different minimax class for each connected component Ω of the intersection $Q_0 \cap Q_1$ is that, taking the infimum over all k_Ω , one gets an a priori better critical value and, hence, a sharper result.*

6. COUNTEREXAMPLES

Throughout this section Σ will be a closed connected orientable surface and $\tilde{\Sigma}$ will be its universal cover. Consider the hyperbolic plane

$$\mathbb{H} := \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0 \right\}$$

endowed with the Riemannian metric

$$g_{(x_1, x_2)} := \frac{1}{x_2^2} (dx_1^2 + dx_2^2). \quad (6.1)$$

We refer to [9] for generalities and properties of (\mathbb{H}, g) . We define

$$L : T\mathbb{H} \longrightarrow \mathbb{R}, \quad L(q, v) = \frac{1}{2} |v|_q^2 + \vartheta_q(v); \quad (6.2)$$

where $\vartheta_{(x_1, x_2)} = \frac{dx_1}{x_2}$ is the “canonical primitive” of the standard area form

$$\sigma = \frac{1}{x_2^2} dx_1 \wedge dx_2.$$

It is well-known that $c(L) = \frac{1}{2}$ (c.f. [13, Section 5.2]). In fact, the Hamiltonian associated with L is

$$H(q, p) = \frac{1}{2} |p - \vartheta_q|^2$$

and hence (3.6) implies that

$$c(L) = \inf_{u \in C^\infty(\mathbb{H})} \sup_{q \in \mathbb{H}} \frac{1}{2} |d_q u - \vartheta_q|^2 \leq \frac{1}{2},$$

as $|\vartheta_q| \equiv 1$. Computing the $(L + k)$ -action of the clockwise arc-length parametrization γ_r of a (hyperbolic) circle with radius r yields the opposite inequality. Indeed, using

$$l(\gamma_r) = 2\pi \sinh r, \quad \text{Area}(D_r) = 2\pi (\cosh r - 1),$$

we readily compute for the action of γ_r

$$S_k(\gamma_r) = \int_0^{l(\gamma_r)} \left[\frac{1}{2} |\dot{\gamma}_r(t)|^2 + k \right] dt + \int_{\gamma_r} \vartheta = \pi \left(k - \frac{1}{2} \right) e^r + f(r),$$

with $f(r)$ uniformly bounded function of r . It follows that, for every $k < \frac{1}{2}$

$$S_k(\gamma_r) \longrightarrow -\infty$$

as r goes to infinity, thus showing that $c(L) \geq \frac{1}{2}$. The restriction of the Euler-Lagrange flow to the energy level set $E^{-1}(\frac{1}{2})$ is the celebrated *horocycle flow* of Hedlund (cf. [9] and [20]). Its peculiarity relies on the fact that, once projected to a compact quotient of \mathbb{H} , it becomes *minimal*, meaning that every orbit is dense. For $k < \frac{1}{2}$, the Euler-Lagrange flow on $E^{-1}(k)$ is periodic and the projections of the orbits to \mathbb{H} describe circles with hyperbolic (thus, euclidean) radius going to zero as $k \rightarrow 0$.

Orbits connecting two points. In this subsection we show an approximated counterexample to Contreras’ result [11] about the existence of Euler-Lagrange orbits connecting two points $q_0 \neq q_1 \in M$. Strictly speaking, for every $\epsilon > 0$, we embed the flow of the Lagrangian in (6.2) into any surface Σ in a suitable fashion. If the points q_0 and q_1 are chosen properly, then they cannot be connected by orbits with energy less than $c_u - \epsilon$.

Thus, consider the Euler-Lagrange flow on $T\mathbb{H}$ associated to the Lagrangian in (6.2) and fix $\epsilon > 0$, $q_0 \in \mathbb{H}$. We know that, for every $k < \frac{1}{2}$, the restriction of the Euler-Lagrange flow to $E^{-1}(k)$ is

periodic and orbits describe hyperbolic (hence, euclidean) circles with the same hyperbolic radius. If we denote by $\rho(q_0, v)$ the euclidean radius of the (projection of the unique) Euler-Lagrange orbit through (q_0, v) , then we readily have

$$\rho := \max_{k \leq \frac{1}{2} - \epsilon} \max_{|v|=k} \rho(q_0, v) < \infty.$$

Let now $B_1 \subseteq B_2 \subseteq B_3$ be open connected sets containing q_0 such that all Euler-Lagrange orbits with energy less than $\frac{1}{2} - \epsilon$ starting from q_0 are entirely contained in B_1 . We extend $\vartheta|_{B_1}$ to be equal to zero outside B_2 using a suitable cut-off function and embed B_3 in Σ . The embedding induces a Riemannian metric on a subset \mathcal{U} of Σ which can be extended to a metric on the whole Σ and also a 1-form on Σ obtained simply by setting the pull-back of ϑ to be zero outside \mathcal{U} .

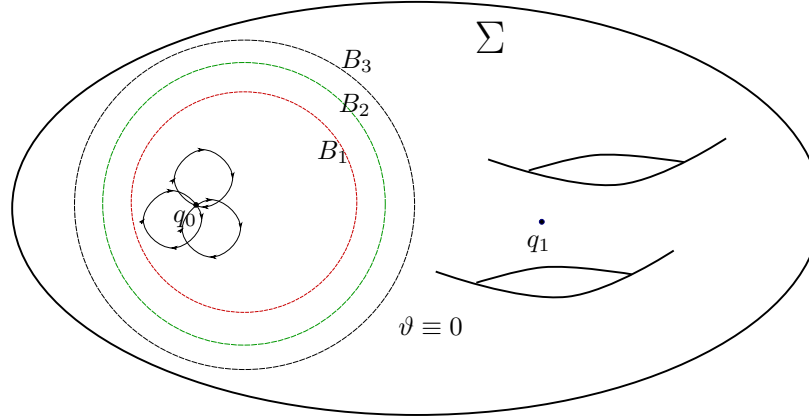


Figure 3. An approximated counterexample to Contreras' result.

We denote the metric, the 1-form and the point on Σ given by the embedding again with g, ϑ, q_0 respectively and define the magnetic Lagrangian

$$L_\epsilon : T\Sigma \rightarrow \mathbb{R}, \quad L(q, v) = \frac{1}{2} |v|^2 + \vartheta_q(v).$$

If we now consider $q_1 \in \Sigma \setminus \mathcal{U}$, then by construction there are no Euler-Lagrange orbits with energy k connecting q_0 with q_1 for every $k < \frac{1}{2} - \epsilon$. This automatically implies that $c_u(L_\epsilon) \geq \frac{1}{2} - \epsilon$ by Contreras' result. At the same time, by (3.6),

$$c_u(L_\epsilon) = \inf_{u \in C^\infty(\tilde{\Sigma})} \sup_{\tilde{q} \in \tilde{\Sigma}} \frac{1}{2} |d_{\tilde{q}}u - \tilde{\vartheta}_{\tilde{q}}|^2 \leq \frac{1}{2},$$

where $\tilde{\vartheta}$ denotes the lift of ϑ to $\tilde{\Sigma}$, since $|\vartheta_q| \leq 1$ for every $q \in \Sigma$.

Supercritical energies. In this subsection we prove Theorem 2. We also show that in general one cannot expect the existence of local minimizers (necessarily not global) for \mathbb{A}_k in the energy range $(c_u(L), c(L; Q_0, Q_1))$, even if the configuration space is a surface; this is in sharp contrast with what happens in the case of periodic orbits (see e.g. [2] or [14]).

The example constructed in the previous subsection suggests that below $c_u(L)$ we might not expect to find orbits connecting two given disjoint submanifolds. However in the example we gave above we had $c_u(L) = c(L; Q_0, Q_1)$. The natural question is now to study what happens for

$$k \in (c_u(L), c(L; Q_0, Q_1)).$$

In fact, for every energy in this range every point q_0 of Q_0 can be joined with every point q_1 of Q_1 . Namely, for every k in this energy range, the free-time action functional \mathbb{A}_k on the space \mathcal{M}_q of H^1 -paths from q_0 to q_1 is bounded from below and satisfies the Palais-Smale condition; it follows that \mathbb{A}_k has a global minimizer on each connected component of \mathcal{M}_q , which therefore corresponds to an Euler-Lagrange orbit from q_0 to q_1 . What is not clear is whether such an orbit connecting q_0 to q_1 also satisfies the conormal boundary conditions. Actually, it does not need to, as we now

show. Namely, we exhibit an example of a magnetic Lagrangian and disjoint submanifolds Q_0, Q_1 such that $c_u(L) < c(L; Q_0, Q_1)$ and for every $k < c(L; Q_0, Q_1)$ there are no orbits satisfying the conormal boundary conditions. We shall start producing a situation where

$$0 < c_u(L) < c(L);$$

this is inspired by the construction in [23]. Think of \mathbb{T}^2 as the square $[0, 1]^2$ in \mathbb{R}^2 with identified sides and equipped with the euclidean metric and consider the magnetic Lagrangian

$$L : T\mathbb{T}^2 \longrightarrow \mathbb{R}, \quad L(q, v) = \frac{1}{2}|v|^2 + \psi(y)v_x, \quad (6.3)$$

where $q = (x, y)$, $v = (v_x, v_y)$ and $\psi : [0, 1] \rightarrow [0, 1]$ is a smooth cut-off function compactly supported in $(0, 1)$ with $\psi \leq 1$, $\psi(\frac{1}{2}) = 1$, and $\psi' \geq 0$ on $[0, \frac{1}{2}]$, $\psi' \leq 0$ on $[\frac{1}{2}, 1]$.

The Lagrangian in (6.3) is a magnetic Lagrangian with magnetic 1-form $\vartheta_q(\cdot) = \psi(y)dx$. Clearly $|\vartheta_q| = |\psi(y)|$ for every $q = (x, y)$ and hence

$$c(L) = \inf_{u \in C^\infty(\mathbb{T}^2)} \max_{q \in \mathbb{T}^2} \frac{1}{2} |d_q u - \vartheta_q|^2 \leq \frac{1}{2}.$$

Conversely, consider the path $a : [0, 1] \rightarrow \mathbb{R}^2$, $a(t) = (1 - t, \frac{1}{2})$; it is clear that a is closed as a path in \mathbb{T}^2 . We now readily compute for $k > 0$

$$S_k(a) = \int_0^1 \left(\frac{1}{2} |\dot{a}(t)|^2 + \psi(a(t))\dot{a}_x(t) + k \right) dt = \int_0^1 \left(\frac{1}{2} |1|^2 - 1 + k \right) dt = k - \frac{1}{2},$$

which is negative for every $k < \frac{1}{2}$. We may then conclude that $c(L) = \frac{1}{2}$.

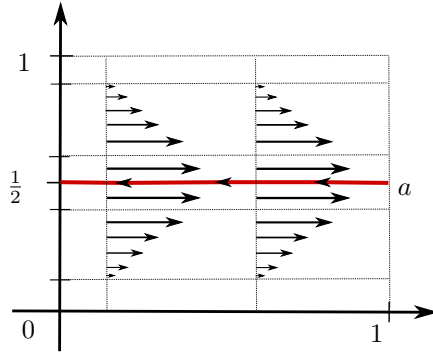


Figure 4. The Mañé critical value $c(L)$ of the Lagrangian L in (6.3) equals $\frac{1}{2}$.

Again, by the Hamiltonian characterization of the Mañé critical value we have

$$c_u(L) = \inf_{u \in C^\infty(\mathbb{R}^2)} \sup_{q \in \mathbb{R}^2} \frac{1}{2} |d_q u - \vartheta_q|^2 \leq \frac{1}{8}$$

as one gets by choosing $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u(x, y) = \frac{x}{2}$:

$$\frac{1}{2} \left| \frac{1}{2} dx - \psi(y) dx \right|^2 = \frac{1}{2} \left| \frac{1}{2} - \psi(y) \right|^2 \leq \frac{1}{8}, \quad \forall (x, y) \in \mathbb{R}^2,$$

since $0 \leq \psi(y) \leq 1$ for every y . On the other hand, for every $n \in \mathbb{N}$ consider the contractible loop $\alpha_n := e \# d^n \# c \# b^n$ obtained concatenating the following paths b, c, d, e with constant speed $\sqrt{2k}$:

$$\begin{aligned} b &: [0, \frac{1}{\sqrt{2k}}] \rightarrow \mathbb{R}^2, \quad t \mapsto (1 - \sqrt{2k}t, \frac{1}{2}) \\ c &: [0, \frac{1}{2\sqrt{2k}}] \rightarrow \mathbb{R}^2, \quad t \mapsto (0, \frac{1}{2} - \sqrt{2k}t) \\ d &: [0, \frac{1}{\sqrt{2k}}] \rightarrow \mathbb{R}^2, \quad t \mapsto (\sqrt{2k}t, 0) \\ e &: [0, \frac{1}{2\sqrt{2k}}] \rightarrow \mathbb{R}^2, \quad t \mapsto (1, \sqrt{2k}t). \end{aligned}$$

A straightforward computation shows that, for $k < \frac{1}{8}$,

$$\begin{aligned} S_k(\alpha_n) &= n \cdot S_k(b) + S_k(c) + n \cdot S_k(d) + S_k(e) \\ &= \frac{n}{\sqrt{2k}}(2k - \sqrt{2k}) + \frac{2k}{2\sqrt{2k}} + \frac{n}{\sqrt{2k}}(2k) + \frac{2k}{2\sqrt{2k}} \\ &= n(2\sqrt{2k} - 1) + \sqrt{2k} \end{aligned}$$

goes to $-\infty$ as $n \rightarrow +\infty$. It follows that $c_u(L) = \frac{1}{8}$.

We pick now Q_0 to be any point in \mathbb{T}^2 , for instance $(\frac{1}{2}, 0)$ and Q_1 to be the circle $\{y = \frac{1}{2}\}$; by construction we have

$$c(L; Q_0, Q_1) = c(L) = \frac{1}{2}.$$

Since $\pi_0(\mathcal{M}_Q) \cong \mathbb{Z}$, Theorem 1 implies that for every $k > \frac{1}{2}$ there are infinitely many Euler-Lagrange orbits with energy k satisfying the conormal boundary conditions. Namely, there is one such orbit for every connected component of \mathcal{M}_Q , which is in addition a global minimizer of \mathbb{A}_k on its connected component. Furthermore, for every $k \in (\frac{1}{8}, \frac{1}{2})$ and any point $q_1 \in Q_1$ there are infinitely many Euler-Lagrange orbits with energy k joining $q_0 = Q_0$ with q_1 . However, none of these can satisfy the conormal boundary conditions for Q_1 since by the obstruction (1.5) this is possible only above energy $\frac{1}{2}$ (observe indeed that in this example $k(L; Q_0, Q_1) = c(L; Q_0, Q_1) = \frac{1}{2}$).

The same counterexample holds clearly for every point of the form $q_0 = (\frac{1}{2}, h)$ for every $h > 0$, in particular showing that we might not expect to find Euler-Lagrange orbits with energy less than $c(L; Q_0, Q_1)$ satisfying the conormal boundary conditions, even if the two submanifolds are “close” to each other. Notice that this example for $q_0 = (\frac{1}{2}, \frac{1}{2})$ is not in contradiction with Theorem 3, since in this case we have

$$k_{Q_0 \cap Q_1} = c(L; Q_0, Q_1).$$

The Lagrangian in (6.3) gives also a sharp counterexample to Contreras’ result. Namely, if $q_0 = (\frac{1}{2}, 0)$ and $q_1 = (\frac{1}{2}, \frac{1}{2})$, then there are no Euler-Lagrange orbits connecting the two points for $k \leq \frac{1}{8} = c_u(L)$. This can be seen as follows: The function $I : T\mathbb{T}^2 \rightarrow \mathbb{R}$ given by $I(q, v) := v_x + \psi(y)$ is an integral of the motion. Computing $I(q_0, v)$ and $I(q_1, v)$ for every $v \in \mathbb{R}^2$ with $|v| = \sqrt{2k}$ yields

$$I(q_0, v) = v_x \in [-\sqrt{2k}, \sqrt{2k}], \quad I(q_1, v) = v_x + 1 \in [1 - \sqrt{2k}, 1 + \sqrt{2k}].$$

Since the two intervals are disjoint for $k < \frac{1}{8}$, there are no orbits connecting the two points with energy $k < \frac{1}{8}$. Moreover, for $k = \frac{1}{8}$, the only possible Euler-Lagrange orbit connecting the two points must start and end parallel to the x -axis. However, this is not possible because the orbits starting from q_0 or q_1 with tangent vector parallel to the x -axis are periodic orbits parallel to the x -axis. Note moreover that, for $k = \frac{1}{8}$, there are heteroclinic orbits connecting the two periodic orbits $\{y = 0\}$ and $\{y = \frac{1}{2}\}$, provided that the function ψ has non-degenerate minimum and maximum at $y = 0$ and $y = \frac{1}{2}$, respectively.

We finally observe that the Lagrangian in (6.3) can also be used to give an example in which the interval $(c(L; Q_0, Q_1), k_{\mathcal{N}}(L))$ as in the statement 2-(a) of Theorem 1 is non-empty. Just consider as Q_0 and Q_1 two small (contractible) intersecting circles with center on $\{y = \frac{1}{2}\}$.

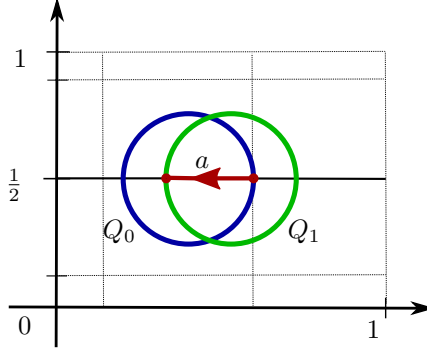


Figure 5. An example where $c(L; Q_0, Q_1) < k_{\mathcal{N}}(L)$.

One readily sees that for such a choice of submanifolds there holds

$$c(L; Q_0, Q_1) = c_u(L) < c(L) = k_{\mathcal{N}}(L) = \frac{1}{2}.$$

Indeed, the path $a : J \rightarrow \mathbb{T}^2$ with constant speed 1 depicted in Figure 5 lies in the connected component \mathcal{N} containing the constant paths and satisfies $\mathbb{A}_k(a) = |J|(k - \frac{1}{2})$; this shows that $k_{\mathcal{N}}(L) \geq \frac{1}{2}$ and hence automatically $k_{\mathcal{N}}(L) = \frac{1}{2}$, as on the other hand by (3.11) we have

$$k_{\mathcal{N}}(L) \leq e_0(L) + \frac{\|\vartheta\|_{\infty}^2}{4a} = \frac{\|\vartheta\|_{\infty}^2}{2} = \frac{1}{2},$$

since $e_0(L) = 0$ and $a = \frac{1}{2}$ for magnetic Lagrangians, and $\|\vartheta\|_{\infty} = 1$.

Subcritical energies. In this subsection we prove Theorem 4 by constructing an example of magnetic Lagrangian $L : T\Sigma \rightarrow \mathbb{R}$ and intersecting submanifolds Q_0, Q_1 such that $c_u(L) \in [\frac{3}{2}, 2]$, $k_{Q_0 \cap Q_1} = \frac{1}{2}$, $k(L; Q_0, Q_1) = 0$, and there are no connecting orbits with energy less than $\frac{1}{2}$.

We start considering the 1-form 2ϑ on \mathbb{H} and the associated magnetic Lagrangian $L(q, v) = \frac{1}{2}|v|_q^2 + 2\vartheta_q(v)$; here ϑ is the canonical primitive of the standard area form in \mathbb{H} . It is easy to see that $c_u(L) = 2$. Now let Q_0 be a rounded up rectangle in \mathbb{H} ; to fix the notation say that the vertical sides of Q_0 have $x = a$ and $x = b$ respectively. Moreover, fix $c < d$, with $c, d \in [a, b]$, such that all orbits with energy $k \leq \frac{1}{2}$ and starting on the vertical sides stay in the region $\{x \notin [c, d]\}$. Up to increasing the length of the horizontal sides of Q_0 one sees that c and d actually exist. Take now a proper subinterval $[e, f] \subset [c, d]$ and let $\varphi : \mathbb{R} \rightarrow [1, 2]$ be a smooth function with the following properties:

- $\varphi \equiv 2$ on $\mathbb{R} \setminus (c, d)$,
- $\varphi \equiv 1$ on $[e, f]$,
- φ decreasing on $[c, e]$ and increasing on $[f, d]$.

Now consider $\vartheta' = \varphi(x) \cdot \vartheta$, the associated magnetic Lagrangian L' and set Q_1 to be any circle intersecting Q_0 and contained in the region $\{x \in [e, f]\}$ (see Figure 6 below). Take $C \subset C' \subset \mathbb{H}$ sufficiently large compact sets as in Figure 6 and such that C contains loops with negative $(L' + \frac{3}{2})$ -action (observe that this is possible since $c_u(L') = 2$). Finally, using a suitable cut-off function, define from ϑ' a new 1-form ϑ'' on \mathbb{H} such that $\vartheta'' \equiv \vartheta'$ on C and $\vartheta'' \equiv 0$ outside C' , and consider the associated magnetic Lagrangian L'' .

Embedding this example into any surface Σ as done at the beginning of this section yields a magnetic Lagrangian $L : T\Sigma \rightarrow \mathbb{R}$ and intersecting submanifolds Q_0, Q_1 such that:

- $c_u(L) \in (\frac{3}{2}, 2]$ and $k_{Q_0 \cap Q_1} = \frac{1}{2}$.
- For almost every $k \in (\frac{1}{2}, c_u(L))$ there is a connecting orbit with energy k by Theorem 3.
- For every $k < \frac{1}{2}$ there are no connecting orbits with energy k . Indeed, since $|\mathcal{P}_0 w_q| \geq 1$ on the horizontal edges of Q_0 , by the obstruction (1.5) such connecting orbits should start from the vertical edges of Q_0 . However, by construction the orbits starting from the vertical

edges of Q_0 do not intersect the region $\{x \in [e, f]\}$ and, hence, they cannot be connecting orbits.

- $k(L; Q_0, Q_1) = 0$. In particular, the condition $k > k(L; Q_0, Q_1)$ is not sufficient to guarantee the existence of connecting orbits.

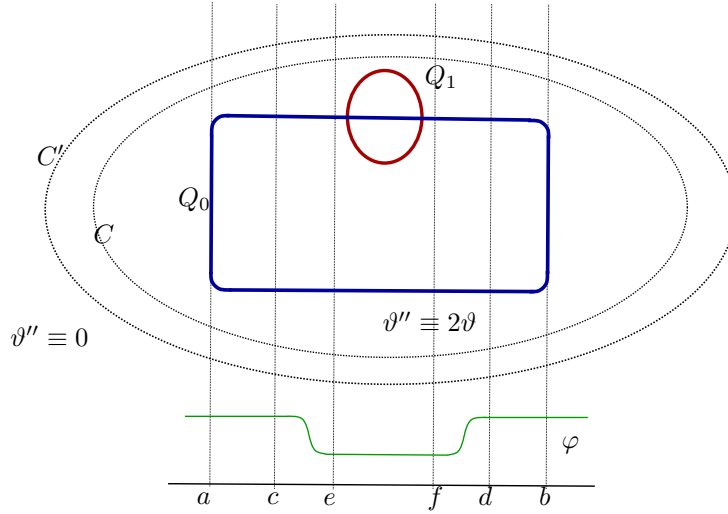


Figure 6. Theorem 3 is sharp.

Acknowledgments. I warmly thank Alberto Abbondandolo for drawing my attention on the subject. I am also grateful to the anonymous referee for his very careful reading of the draft and for his precious suggestions which helped me to improve essentially the paper. The author is partially supported by the DFG grant AB 360/2-1 “Periodic orbits of conservative systems below the Mañé critical energy value”.

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